*Adv. Appl. Prob.* **35,** 504–531 (2003)
*Printed in Northern Ireland*


# FIRST PASSAGE TIMES OF A JUMP DIFFUSION PROCESS

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**Abstract**

This paper studies the first passage times to flat boundaries for a double exponential jump diffusion process, which consists of a continuous part driven by a Brownian motion and a jump part with jump sizes having a double exponential distribution. Explicit solutions of the Laplace transforms, of both the distribution of the first passage times and the joint distribution of the process and its running maxima, are obtained. Because of the overshoot problems associated with general jump diffusion processes, the double exponential jump diffusion process offers a rare case in which analytical solutions for the first passage times are feasible. In addition, it leads to several interesting probabilistic results. Numerical examples are also given. The finance applications include pricing barrier and lookback options.

*Keywords:* Renewal theory; martingale; differential equation; integral equation; infinitesimal generator; marked point process; Lévy process; Gaver–Stehfest algorithm

AMS 2000 Subject Classification: Primary 60J75; 44A10
Secondary 60J27

## 1. Introduction

### 1.1. Background

Jump diffusion processes are processes of the form

$$X_t = \sigma W_t + \mu t + \sum_{i=1}^{N_t} Y_i; \qquad X_0 \equiv 0. \tag{1.1}$$

Here $\{W_t; t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$, $\{N_t; t \geq 0\}$ is a Poisson process with rate $\lambda$, constants $\mu$ and $\sigma > 0$ are the drift and volatility of the diffusion part respectively, and the jump sizes $\{Y_1, Y_2, \dots\}$ are independent and identically distributed (i.i.d.) random variables. We also assume that the random processes $\{W_t; t \geq 0\}$, $\{N_t; t \geq 0\}$, and random variables $\{Y_1, Y_2, \dots\}$ are independent. Note that the jump part, $\sum_{i=1}^{N_t} Y_i$, is a special case of the so-called marked point processes; further background on marked point processes can be found, for example, in [9], [15]. The processes in (1.1) have been given different names in the literature, and they are indeed special cases of Lévy processes; see e.g. [6], [26] for background on Lévy processes.

The jump diffusion processes are widely used, for example, in finance to model asset (stock, bond, currency, etc.) prices. Two examples are the normal jump diffusion process where $Y$

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has a normal distribution (e.g. [20]) and the double exponential jump diffusion process where $Y$ has a double exponential distribution (e.g. [18]), i.e. the common density of $Y$ is given by

$$f_Y(y) = p \cdot \eta_1 \mathrm{e}^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + q \cdot \eta_2 \mathrm{e}^{\eta_2 y} \mathbf{1}_{\{y < 0\}},$$

where $p, q \geq 0$ are constants, $p + q = 1$, and $\eta_1, \eta_2 > 0$. Note that the means of the two exponential distributions are $1/\eta_1$ and $1/\eta_2$ respectively.

This paper focuses on the first passage times of the double exponential jump diffusion process:

$$\tau_b := \inf\{t \geq 0;\ X_t \geq b\}, \qquad b > 0,$$

where $X_{\tau_b} := \limsup_{t \to \infty} X_t$ on the set $\{\tau_b = \infty\}$. The main problems studied include the distribution of the first passage time

$$\mathrm{P}(\tau_b \leq t) = \mathrm{P}\Big(\max_{0 \leq s \leq t} X_s \geq b\Big) \tag{1.2}$$

for all $t > 0$, the joint distribution between the first passage time and the terminal value

$$\mathrm{P}(\tau_b \leq t,\ X_t \geq a), \tag{1.3}$$

and other related quantities.

There are three reasons why these problems are interesting. First, from a purely probabilistic point of view, the double exponential jump diffusion process offers a rare case in which analytical solutions of the first passage times are feasible. Because of the jump part, when a jump diffusion process crosses the boundary level $b$, sometimes it may incur an ‘overshoot’ over the boundary. In general, the distribution of the overshoot is not known analytically, thus making it impossible to get closed-form solutions of the distribution of the first passage times. However, if the jump sizes, the $Y_i$, have an exponential-type distribution, then the overshoot problems can be solved analytically, thanks to the memoryless property associated with the exponential distribution. See [27, Chapter 8] and [29] for some detailed discussions on overshoot problems, including the ‘ladder variables’ for two-sided distributions which make the overshoot problems more complicated.

Second, the study leads to several interesting probabilistic results. (i) Although the exponential random variables have memoryless properties, the first passage times and the overshoot are dependent, despite the fact that the two are conditionally independent given that the overshoot is bigger than 0. (ii) The renewal-type integral equations, which are used frequently in studying first passage times, may not lead to unique solutions for the problems, because the boundary conditions are difficult to determine; see Section 3.3. Instead, our approach based on differential equations and martingales can circumvent this problem of uniqueness.

Third, from the applied probability point of view, the results of this paper are useful in option pricing. Brownian motion and the normal distribution have been widely used, for example, in the Black–Scholes option pricing framework, to study the return of assets. However, two empirical puzzles have recently received a great deal of attention, namely the leptokurtic feature, meaning that the return distribution of assets may have a higher peak and two (asymmetric) heavier tails than those of the normal distribution, and an abnormality called ‘volatility smile’ in option pricing. Many studies have been conducted to modify the Black–Scholes models in order to explain the two puzzles; see the textbooks [10], [13] for more details. An immediate problem with these different models is that it may be difficult to obtain analytical solutions for the prices of options, especially for those of the popular path-dependent options, such as barrier and lookback options. To get analytically tractable models, and to incorporate both the leptokurtic feature and the 'volatility smile', the double exponential jump diffusion model is proposed in [18]; see also [12] for pricing of interest rate derivatives under such a model and more background about general jump diffusion models.

The explicit calculation of (1.2) and (1.3) or, more precisely, their Laplace transforms, can be used to get closed-form solutions for pricing barrier and lookback options under the double exponential jump diffusion model. The details of its finance applications, being too long to be included here, are reported in [19]. Using a Laplace inverse algorithm (the Gaver–Stehfest algorithm), both (1.2) and (1.3) can then be computed very quickly.

### 1.2. Intuition

Without the jump part, the process simply becomes a Brownian motion with drift $\mu$. These distributions of the first passage times can be obtained either by a combination of a change of measure (Girsanov theorem) and the reflection principle, or by calculating the Laplace transforms via some appropriate martingales and the optional sampling theorem. Details of both methods can be found in many classical textbooks on stochastic analysis, e.g. [16], [17].

With the jump part, however, it is very difficult to study the first passage times for general jump diffusion processes. When a jump diffusion process crosses boundary level $b$, sometimes it hits the boundary exactly and sometimes it incurs an 'overshoot', $X_{\tau_b} - b$, over the boundary. The overshoot presents several problems if we want to compute the distribution of the first passage times analytically.

(i) We need to get the exact distribution of the overshoot, $X_{\tau_b} - b$; particularly, $\mathrm{P}(X_{\tau_b} - b = 0)$ and $\mathrm{P}(X_{\tau_b} - b > x)$ for $x > 0$. This is possible if the jump size $Y$ has an exponential-type distribution, thanks to the memoryless property of the exponential distribution.

(ii) We need to know the dependent structure between the overshoot, $X_{\tau_b} - b$, and the first passage time $\tau_b$. Given that the overshoot is bigger than 0, the two random variables are conditionally independent and $X_{\tau_b} - b$ is conditionally memoryless if the jump size $Y$ has an exponential-type distribution. This conditionally independent and conditionally memoryless structure seems to be peculiar to the exponential distribution, and does not hold for general distributions.

(iii) If we want to use the reflection principle to study the first passage times, the dependent structure between the overshoot and the terminal value $X_t$ is also needed. To the best of the authors' knowledge, this is not known even for the double exponential jump diffusion process.

Consequently, we can derive closed-form solutions for the Laplace transforms of the first passage times for the double exponential jump diffusion process, yet cannot give more explicit calculations beyond that, as the correlation between $X_t$ and $X_{\tau_b} - b$ is not available. However, for other jump diffusion processes, even analytical forms of the Laplace transforms seem to be quite difficult, if not impossible, to obtain.

To compute the Laplace transform of $\mathrm{P}(\tau_b \leq t)$, we use both martingale and differential equations. There are two other possible approaches: renewal-type integral equations and Wiener–Hopf factorization. Renewal-type integral equations are frequently used in the actuarial science literature (see, for example, [11] and references therein) to study first passage times. However, the renewal equation does not lead to a unique solution (see Section 3.3 for details), and thus would not solve the problem in our case.

Wiener–Hopf factorization and related fluctuation identities [6], [7], [21], [25], [26] have also been widely used to study the first passage times for Lévy processes (note that the double exponential jump diffusion process is a special case of Lévy processes). Many such studies focus on one-sided jumps; e.g. [24]. However, because of the one-sided jumps, the 'overshoot' problems are avoided, as either the jumps are in the opposite direction to the barrier crossing or there is no 'ladder variable' problem for the one-sided jumps. The Wiener–Hopf factorization for general jump diffusion processes with two-sided jumps is discussed in [8]. In general, explicit calculation of the Wiener–Hopf factorization is difficult. Because of the special structure of the exponential distribution, especially due to its memoryless property, we can solve the first passage time problems explicitly; in some sense this also suggests, though indirectly, that the Wiener–Hopf factorization could be performed explicitly in the case of double exponential jump diffusion processes.

An outline of the paper is as follows. Section 2 gives some preliminary results. Section 3 presents the computation of the Laplace transform of the first passage times, its immediate corollaries, and its connection with the integral equation approach. The joint distribution of the jump diffusion process and its running maxima is considered in Section 4. Inversion of Laplace transforms and numerical examples are given in Section 5. Some proofs and technical details are deferred to appendices to ease exposition.

## 2. Preliminary results

The infinitesimal generator of the jump diffusion process (1.1) is given by

$$\mathcal{L}u(x) = \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_Y(y)\,\mathrm{d}y$$

for all twice continuously differentiable functions $u(x)$. In addition, suppose that $\theta \in (-\eta_2, \eta_1)$. The moment generating function of jump size $Y$ is given by

$$\mathrm{E}[\mathrm{e}^{\theta Y}] = \frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta},$$

from which the moment generating function of $X_t$ can be obtained as

$$\phi(\theta, t) := \mathrm{E}[\mathrm{e}^{\theta X_t}] = \exp\{G(\theta)t\},$$

where the function $G(\cdot)$ is defined as

$$G(x) := x\mu + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right).$$

**Lemma 2.1.** *The equation*

$$G(x) = \alpha \quad \text{for all } \alpha > 0$$

*has exactly four roots:* $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$, *where*

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty, \qquad 0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty.$$

*In addition, let the overall drift of the jump diffusion process be*

$$\bar{u} := \mu + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right).$$

*Then, as* $\alpha \to 0$,

$$\beta_{1,\alpha} \to \begin{cases} 0 & \text{if } \bar{u} \geq 0, \\ \beta_1^* & \text{if } \bar{u} < 0, \end{cases} \quad \text{and} \quad \beta_{2,\alpha} \to \beta_2^*,$$

*where* $\beta_1^*$ *and* $\beta_2^*$ *are defined as the unique roots*

$$G(\beta_1^*) = 0, \qquad G(\beta_2^*) = 0, \qquad 0 < \beta_1^* < \eta_1 < \beta_2^* < \infty.$$

*Proof.* Since $G(\beta)$ is a convex function on the interval $(-\eta_2, \eta_1)$ with $G(0) = \lambda(p+q-1) = 0$ and $G(\eta_1-) = +\infty$, $G(-\eta_2+) = +\infty$, there is exactly one root $\beta_{1,\alpha}$ for $G(x) = \alpha$ on the interval $(0, \eta_1)$, and another one on the interval $(-\eta_2, 0)$. Furthermore, since $G(\eta_1+) = -\infty$ and $G(+\infty) = \infty$, there is at least one root on $(\eta_1, \infty)$. Similarly, there is at least one root on $(-\infty, -\eta_2)$, as $G(-\infty) = \infty$ and $G(-\eta_2-) = -\infty$. But the equation $G(\beta) = \alpha$ is actually a polynomial equation with degree four; therefore, it can have at most four real roots. It follows that, on each interval, $(-\infty, -\eta_2)$ and $(\eta_1, \infty)$, there is exactly one root.

The limiting results when $\alpha \to 0$ follow easily once we note that $G'(0) = \bar{u}$.

The following result shows that the memoryless property of the random walk of exponential random variables leads to the conditional memoryless property of the jump diffusion process.

**Proposition 2.1.** (Conditional memorylessness and conditional independence.) *For any* $x > 0$,

$$P(\tau_b \leq t, X_{\tau_b} - b \geq x) = e^{-\eta_1 x} P(\tau_b \leq t, X_{\tau_b} - b > 0), \tag{2.1}$$

$$P(X_{\tau_b} - b \geq x \mid X_{\tau_b} - b > 0) = e^{-\eta_1 x}. \tag{2.2}$$

*Furthermore, conditional on* $X_{\tau_b} - b > 0$, *the stopping time* $\tau_b$ *and the overshoot* $X_{\tau_b} - b$ *are independent; more precisely, for any* $x > 0$,

$$\begin{aligned} &P(\tau_b \leq t, X_{\tau_b} - b \geq x \mid X_{\tau_b} - b > 0) \\ &\quad = P(\tau_b \leq t \mid X_{\tau_b} - b > 0)\, P(X_{\tau_b} - b \geq x \mid X_{\tau_b} - b > 0). \end{aligned} \tag{2.3}$$

*Proof.* We only need to show that (2.1) holds. The equality (2.2) follows readily by letting $t \to \infty$ and observing that, on the set $\{X_{\tau_b} > b\}$, the hitting time $\tau_b$ is finite by definition; and (2.3) also holds since

$$\begin{aligned} &P(\tau_b \leq t, X_{\tau_b} - b \geq x \mid X_{\tau_b} - b > 0) \\ &\quad = \frac{P(\tau_b \leq t, X_{\tau_b} - b \geq x)}{P(X_{\tau_b} - b > 0)} \\ &\quad = e^{-\eta_1 x} \frac{P(\tau_b \leq t, X_{\tau_b} - b > 0)}{P(X_{\tau_b} - b > 0)} \\ &\quad = P(X_{\tau_b} - b \geq x \mid X_{\tau_b} - b > 0)\, P(\tau_b \leq t \mid X_{\tau_b} - b > 0). \end{aligned}$$

Denote by $T_1, T_2, \ldots$ the arrival times of the Poisson process $N$. It follows that

$$P(\tau_b \leq t, X_{\tau_b} - b \geq x) = \sum_{n=1}^{\infty} P(T_n = \tau_b \leq t, X_{T_n} - b \geq x) =: \sum_{n=1}^{\infty} P_n,$$

as the overshoot inside the probability can only occur during the arrival times of the Poisson process because $x > 0$. However, with $X_0(t) = \sigma W_t + \mu t$, we have

$$\begin{aligned}
P_n &= \mathrm{P}\Big(\max_{0\le s<T_n} X_s < b, X_{T_n} \ge b + x, T_n \le t\Big)\\
&= \mathrm{E}\{\mathrm{P}(X_{T_n} \ge b + x \mid \mathcal{F}_{T_n-}, T_n)\,\mathbf{1}_{\{\max_{0\le s<T_n} X_s<b, T_n\le t\}}\}\\
&= \mathrm{E}\{p \exp\{-\eta_1(b + x - X_0(T_n) - Y_1 - \cdots - Y_{n-1})\}\,\mathbf{1}_{\{\max_{0\le s<T_n} X_s<b, T_n\le t\}}\}\\
&= \mathrm{e}^{-\eta_1 x}\,\mathrm{E}\{p \exp\{-\eta_1(b - X_0(T_n) - Y_1 - \cdots - Y_{n-1})\}\,\mathbf{1}_{\{\max_{0\le s<T_n} X_s<b, T_n\le t\}}\}\\
&= \mathrm{e}^{-\eta_1 x}\,\mathrm{E}\{\mathrm{P}(X_{T_n} > b \mid \mathcal{F}_{T_n-}, T_n)\,\mathbf{1}_{\{\max_{0\le s<T_n} X_s<b, T_n\le t\}}\}\\
&= \mathrm{e}^{-\eta_1 x}\,\mathrm{P}\Big(\max_{0\le s<T_n} X_s < b, X_{T_n} > b, T_n \le t\Big)\\
&= \mathrm{e}^{-\eta_1 x}\,\mathrm{P}(X_{T_n} - b > 0, T_n = \tau_b \le t).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{P}(\tau_b \le t, X_{\tau_b} - b \ge x) &= \sum_{n=1}^{\infty} \mathrm{e}^{-\eta_1 x}\,\mathrm{P}(T_n = \tau_b \le t, X_{\tau_b} - b > 0)\\
&= \mathrm{e}^{-\eta_1 x}\,\mathrm{P}(\tau_b \le t, X_{\tau_b} - b > 0).
\end{aligned}$$

This completes the proof.

**Remark 2.1.** The conditional independence (though not the conditional memorylessness) in Proposition 2.1 actually holds with greater generality; see, for example, [14], [28] and their further generalization [22].

It is easy to verify from Proposition 2.1 that, for any $x > 0$, the following equalities hold:

$$\begin{aligned}
\mathrm{P}(X_{\tau_b} - b \ge x) &= \mathrm{e}^{-\eta_1 x}\,\mathrm{P}(X_{\tau_b} - b > 0),\\
\mathrm{E}(\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}\ge b+x\}}) &= \mathrm{e}^{-\eta_1 x}\,\mathrm{E}(\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}-b>0\}}).
\end{aligned}$$

## 3. Distribution of the first passage times

### 3.1. The Laplace transforms

**Theorem 3.1.** *For any $\alpha \in (0, \infty)$, let $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ be the only two positive roots of the equation*

$$\alpha = G(\beta),$$

*where $0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$. Then we have the following results concerning the Laplace transforms of $\tau_b$ and $X_{\tau_b}$:*

$$\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}] = \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1}\frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}\mathrm{e}^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\eta_1}\frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}\mathrm{e}^{-b\beta_{2,\alpha}}, \tag{3.1}$$

$$\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}-b>y\}}] = \mathrm{e}^{-\eta_1 y}\frac{(\eta_1 - \beta_{1,\alpha})(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})}[\mathrm{e}^{-b\beta_{1,\alpha}} - \mathrm{e}^{-b\beta_{2,\alpha}}] \quad \textit{for all } y \ge 0, \tag{3.2}$$

$$\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}=b\}}] = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}}\mathrm{e}^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}}\mathrm{e}^{-b\beta_{2,\alpha}}. \tag{3.3}$$

*Proof.* Here we focus on the proof for (3.1) and (3.2) since (3.3) follows immediately by taking the difference of (3.1) and (3.2) and by letting $y = 0$. For notational simplicity, we shall write $\beta_i \equiv \beta_{i,\alpha}$, $i = 1, 2$.

We first prove (3.1). For any fixed level $b > 0$, define the function $u$ to be

$$u(x) := \begin{cases} 1, & x \geq b, \\ A_1 \mathrm{e}^{-\beta_1(b-x)} + B_1 \mathrm{e}^{-\beta_2(b-x)}, & x < b, \end{cases}$$

where $A_1$ and $B_1$ are defined to be the two coefficients in front of the exponential terms in (3.1). Clearly, $0 \leq u(x) \leq 1$ for all $x \in (-\infty, \infty)$, because $\beta_1, \beta_2 \geq 0$. Note that, on the set $\{\tau_b < \infty\}$, $u(X_{\tau_b}) = 1$ since $A_1 + B_1 = 1$. Furthermore, the function $u$ is continuous.

Substituting this form of $u$ and doing the integration in two regions, $\int_{-\infty}^{\infty} = \int_{-\infty}^{b-x} + \int_{b-x}^{\infty}$, we have, after some algebra, that, for all $x < b$, $-\alpha u + \mathcal{L}u$ is equal to

$$A_1 \mathrm{e}^{-(b-x)\beta_1} f(\beta_1) + B_1 \mathrm{e}^{-(b-x)\beta_2} f(\beta_2) - \lambda p \mathrm{e}^{-\eta_1(b-x)} \left( A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} - 1 \right), \quad (3.4)$$

where $f(\beta) := G(\beta) - \alpha$. Since

$$f(\beta_1) = f(\beta_2) = 0, \qquad A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} - 1 = 0,$$

we have

$$-\alpha u(x) + \mathcal{L}u(x) = 0 \quad \text{for all } x < b. \quad (3.5)$$

Because the function $u(x)$ is continuous, but not $\mathrm{C}^1$ at $x = b$, we cannot apply Itô's formula directly to the process $\{\mathrm{e}^{-\alpha t} u(X_t);\, t \geq 0\}$. However, it is not difficult to see that there exists a sequence of functions $\{u_n(x);\, n = 1, 2, \ldots\}$ such that: (i) $u_n(x)$ is smooth everywhere, and in particular it belongs to $\mathrm{C}^2$; (ii) $u_n(x) = u(x)$ for all $x \leq b$; (iii) $u_n(x) = 1 = u(x)$ for all $x \geq b + 1/n$; (iv) $0 \leq u_n(x) \leq 2$ for all $x$ and $n$. Clearly, $u_n(x) \to u(x)$ for all $x$.

It follows from a straightforward calculation that, for $x < b$,

$$\begin{aligned}
\mathcal{L}u_n(x) &= \frac{1}{2}\sigma^2 u_n''(x) + \mu u_n'(x) + \lambda \int_{-\infty}^{\infty} [u_n(x+y) - u_n(x)] f_Y(y)\, \mathrm{d}y \\
&= \frac{1}{2}\sigma^2 u_n''(x) + \mu u_n'(x) - \lambda u_n(x) + \lambda \int_{-\infty}^{\infty} u_n(x+y) f_Y(y)\, \mathrm{d}y \\
&= \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) - \lambda u(x) + \lambda \int_{-\infty}^{b-x} u(x+y) f_Y(y)\, \mathrm{d}y \\
&\quad + \lambda \int_{b-x}^{b-x+1/n} u_n(x+y) f_Y(y)\, \mathrm{d}y + \lambda \int_{b-x+1/n}^{\infty} u(x+y) f_Y(y)\, \mathrm{d}y \\
&= \alpha u(x) + \lambda \int_{b-x}^{b-x+1/n} u_n(x+y) f_Y(y)\, \mathrm{d}y - \lambda \int_{b-x}^{b-x+1/n} u(x+y) f_Y(y)\, \mathrm{d}y,
\end{aligned}$$

thanks to (3.5). Since $|u_n - u| \leq 1$ by construction, it follows that

$$\begin{aligned}
|-\alpha u_n(x) + \mathcal{L}u_n(x)| &\leq \lambda p \int_{b-x}^{b-x+1/n} |u_n(x+y) - u(x+y)| \eta_1\, \mathrm{d}y \\
&\leq \frac{\lambda p \eta_1}{n} \to 0 \quad \text{for all } x < b
\end{aligned} \quad (3.6)$$

uniformly in $x$, as $n \to \infty$. Applying the Itô formula for jump processes (see e.g. [23]) to the process $\{e^{-\alpha t} u_n(X_t);\, t \geq 0\}$, we obtain that the process

$$M_t^{(n)} := e^{-\alpha(t\wedge\tau_b)} u_n(X_{t\wedge\tau_b}) - \int_0^{t\wedge\tau_b} e^{-\alpha s}(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s))\, ds, \qquad t \geq 0,$$

is a local martingale starting from $M_0^{(n)} = u_n(0) = u(0)$. However,

$$|M_t^{(n)}| \leq 2 + \frac{\lambda p \eta_1}{n} t \quad \text{for all } t \geq 0,$$

thanks to (3.6). It follows from the dominated convergence theorem that $\{M_t^{(n)};\, t \geq 0\}$ is actually a martingale. In particular,

$$\mathrm{E}\, M_t^{(n)} = \mathrm{E}\biggl[e^{-\alpha(t\wedge\tau_b)} u_n(X_{t\wedge\tau_b}) - \int_0^{t\wedge\tau_b} e^{-\alpha s}(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s))\, ds\biggr] = u(0)$$

for all $t \geq 0$. Letting $n \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \mathrm{E}[e^{-\alpha(t\wedge\tau_b)} u_n(X_{t\wedge\tau_b})] = \mathrm{E}[e^{-\alpha(t\wedge\tau_b)} u(X_{t\wedge\tau_b})]$$

and, thanks to the uniform convergence in (3.6),

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{E}\biggl[\int_0^{t\wedge\tau_b} e^{-\alpha s}(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s))\, ds\biggr] \\
&\quad = \lim_{n\to\infty} \mathrm{E}\biggl[\int_0^{t\wedge\tau_b -} e^{-\alpha s}(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s))\, ds\biggr] \\
&\quad = 0.
\end{aligned}$$

Therefore, for any $t \geq 0$,

$$\begin{aligned}
u(0) &= \mathrm{E}[e^{-\alpha(t\wedge\tau_b)} u(X_{t\wedge\tau_b})] \\
&= \mathrm{E}[e^{-\alpha(t\wedge\tau_b)} u(X_{t\wedge\tau_b})\, \mathbf{1}_{\{\tau_b<\infty\}}] + \mathrm{E}[e^{-\alpha t} u(X_t)\, \mathbf{1}_{\{\tau_b=\infty\}}].
\end{aligned}$$

Now letting $t \to \infty$, we have, thanks to the boundedness of $u$,

$$u(0) = \mathrm{E}[e^{-\alpha\tau_b} u(X_{\tau_b})\, \mathbf{1}_{\{\tau_b<\infty\}}] = \mathrm{E}[e^{-\alpha\tau_b}\, \mathbf{1}_{\{\tau_b<\infty\}}] = \mathrm{E}[e^{-\alpha\tau_b}],$$

as $u(X_{\tau_b}) = 1$ on the set $\{\tau_b < \infty\}$, from which the result follows.

We now prove (3.2); this is very similar to the previous proof and we only give an outline. It suffices to consider the case where $y > 0$, as the case for $y = 0$ follows by letting $y \downarrow 0$. Letting $u(x) := \mathrm{E}^x[e^{-\alpha\tau_b}\, \mathbf{1}_{\{X_{\tau_b} - b > y\}}]$, we expect that $u$ satisfies the equation

$$-\alpha u(x) + \mathcal{L}u(x) = 0$$

for all $x < b$, and $u(x) = 1$ if $x \geq b + y$ while $u(x) = 0$ if $x \in [b, b + y)$. This equation can be explicitly solved. Indeed, consider a solution taking the form

$$u(x) = \begin{cases} 1, & x > b + y, \\ 0, & b < x \leq b + y, \\ A_2 e^{-(b-x)\beta_1} + B_2 e^{-(b-x)\beta_2}, & x \leq b, \end{cases}$$

where the constants $A_2$ and $B_2$ are yet to be determined. Substitute to obtain that

$$\begin{aligned}(-\alpha u + \mathcal{L}u)(x) &= A_2 \mathrm{e}^{-(b-x)\beta_1} f(\beta_1) + B_2 \mathrm{e}^{-(b-x)\beta_2} f(\beta_2) \\ &\quad - \lambda p \mathrm{e}^{-\eta_1(b-x)}\left(\frac{A_2\eta_1}{\eta_1-\beta_1} + \frac{B_2\eta_1}{\eta_1-\beta_2} - \mathrm{e}^{-\eta_1 y}\right) \\ &= 0\end{aligned}$$

for all $x < b$. Since $f(\beta_1) = f(\beta_2) = 0$, it suffices to choose $A_2$ and $B_2$ so that

$$A_2\frac{\eta_1}{\eta_1-\beta_1} + B_2\frac{\eta_1}{\eta_1-\beta_2} = \mathrm{e}^{-\eta_1 y}.$$

Furthermore, the continuity of $u$ at $x = b$ implies that

$$A_2 + B_2 = 0.$$

Solve the equations to obtain $A_2$ and $B_2$ ($A_2 = -B_2$), which are exactly the coefficients in (3.2). A similar argument as before yields that

$$\begin{aligned}u(0) &= \mathrm{E}[\mathrm{e}^{-\alpha\tau_b} u(X_{\tau_b})\,\mathbf{1}_{\{\tau_b<\infty\}}] \\ &= \mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}>b+y\}}\,\mathbf{1}_{\{\tau_b<\infty\}}] \\ &= \mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}-b>y\}}],\end{aligned}$$

as $u(X_{\tau_b}) = \mathbf{1}_{\{X_{\tau_b}>b+y\}}$ on the set $\{\tau_b < \infty\}$, from which the proof is finished.

Note the following Laplace transform, which is convenient for numerical Laplace inversion:

$$\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(\tau_b \le t)\,\mathrm{d}t = \frac{1}{\alpha}\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{d}\,\mathrm{P}(\tau_b \le t) = \frac{1}{\alpha}\,\mathrm{E}(\mathrm{e}^{-\alpha\tau_b}).$$

**Remark 3.1.** The special form of double exponential density functions enables us to explicitly solve the differential–integral equations (3.5) associated with the Laplace transforms, thanks to (3.4). For general jump diffusion processes, however, such explicit solutions will be very difficult, if not impossible, to obtain.

## 3.2. Properties

**Corollary 3.1.** *We have* $\mathrm{P}(\tau_b < \infty) = 1$ *if and only if* $\bar{u} \ge 0$. *Furthermore, if* $\bar{u} \ge 0$, *then*

$$\mathrm{P}(X_{\tau_b} - b > y) = \mathrm{e}^{-\eta_1 y}\frac{\beta_2^* - \eta_1}{\beta_2^*}[1 - \mathrm{e}^{-b\beta_2^*}] \quad \textit{for all } y \ge 0,$$

$$\mathrm{P}(X_{\tau_b} = b) = \frac{\eta_1}{\beta_2^*} + \frac{\beta_2^* - \eta_1}{\beta_2^*}\mathrm{e}^{-b\beta_2^*}.$$

*If* $\bar{u} < 0$, *then*

$$\mathrm{P}(\tau_b < \infty) = \frac{\eta_1 - \beta_1^*}{\eta_1}\frac{\beta_2^*}{\beta_2^* - \beta_1^*}\mathrm{e}^{-b\beta_1^*} + \frac{\beta_2^* - \eta_1}{\eta_1}\frac{\beta_1^*}{\beta_2^* - \beta_1^*}\mathrm{e}^{-b\beta_2^*} < 1,$$

$$\mathrm{P}(\tau_b < \infty, X_{\tau_b} - b > y) = \mathrm{e}^{-\eta_1 y}\frac{(\eta_1 - \beta_1^*)(\beta_2^* - \eta_1)}{\eta_1(\beta_2^* - \beta_1^*)}[\mathrm{e}^{-b\beta_1^*} - \mathrm{e}^{-b\beta_2^*}] \quad \textit{for all } y \ge 0,$$

$$\mathrm{P}(\tau_b < \infty, X_{\tau_b} = b) = \frac{\eta_1 - \beta_1^*}{\beta_2^* - \beta_1^*}\mathrm{e}^{-b\beta_1^*} + \frac{\beta_2^* - \eta_1}{\beta_2^* - \beta_1^*}\mathrm{e}^{-b\beta_2^*}.$$

*Here* $\beta_1^*$ *and* $\beta_2^*$ *are defined as in Lemma 2.1.*

*Proof.* By Lemma 2.1, if $\bar{u} \geq 0$, then $\beta_{1,\alpha} \to 0$ and $\beta_{2,\alpha} \to \beta_2^*$ as $\alpha \to 0$. Thus,

$$\mathrm{P}(\tau_b < \infty) = \lim_{\alpha \to 0} \mathrm{E}[\mathrm{e}^{-\alpha \tau_b}] = 1.$$

If $\bar{u} < 0$, then $\beta_{1,\alpha} \to \beta_1^*$ and $\beta_{2,\alpha} \to \beta_2^*$ as $\alpha \to 0$. The result follows by letting $\alpha \to 0$ in (3.1), (3.2), and (3.3).

**Remark 3.2.** It is surprising to see from Theorem 3.1 and Corollary 3.1 that the first passage time $\tau_b$ and the overshoot $X_{\tau_b} - b$ are dependent, although Proposition 2.1 shows that they are conditionally independent.

**Corollary 3.2.** *The expectation of the first passage time is finite, i.e.* $\mathrm{E}[\tau_b] < \infty$, *if and only if* $\bar{u} > 0$. *Indeed,*

$$\mathrm{E}[\tau_b] = \begin{cases} \dfrac{1}{\bar{u}}\left[b + \dfrac{\beta_2^* - \eta_1}{\eta_1 \beta_2^*}(1 - \mathrm{e}^{-b\beta_2^*})\right], & \text{if } \bar{u} > 0, \\ +\infty, & \text{if } \bar{u} \leq 0. \end{cases}$$

*Furthermore, for* $\bar{u} < 0$, *we have*

$$\mathrm{E}[\tau_b \mathbf{1}_{\{\tau_b < \infty\}}] = C_1 \mathrm{e}^{-b\beta_1^*} + C_2 \mathrm{e}^{-b\beta_2^*} < \infty,$$

*where*

$$C_1 := \frac{1}{\eta_1(\beta_2^* - \beta_1^*)^2}\left[\frac{\beta_2^*(\beta_2^* - \eta_1) + b\beta_2^*(\eta_1 - \beta_1^*)(\beta_2^* - \beta_1^*)}{G'(\beta_1^*)} + \frac{\beta_1^*(\eta_1 - \beta_1^*)}{G'(\beta_2^*)}\right],$$

$$C_2 := \frac{1}{\eta_1(\beta_2^* - \beta_1^*)^2}\left[\frac{\beta_1^*(\beta_1^* - \eta_1) + b\beta_1^*(\eta_1 - \beta_2^*)(\beta_1^* - \beta_2^*)}{G'(\beta_2^*)} + \frac{\beta_2^*(\eta_1 - \beta_2^*)}{G'(\beta_1^*)}\right].$$

*See Lemma 2.1 for the definition of* $(\beta_1^*, \beta_2^*)$.

*Proof.* To ease exposition, we will use $\beta_i$ to denote $\beta_{i,\alpha}$. Since the function $(1/x)(1 - \mathrm{e}^{-x})$ is decreasing for $x \in [0, +\infty)$, it follows that, with probability 1,

$$\frac{1 - \mathrm{e}^{-\alpha \tau_b}}{\alpha} \mathbf{1}_{\{\tau_b < \infty\}} \uparrow \tau_b \mathbf{1}_{\{\tau_b < \infty\}} \quad \text{as } \alpha \downarrow 0.$$

By monotone convergence,

$$\begin{aligned} \mathrm{E}[\tau_b \mathbf{1}_{\{\tau_b < \infty\}}] &= \lim_{\alpha \downarrow 0} \mathrm{E}\left[\frac{1 - \mathrm{e}^{-\alpha \tau_b}}{\alpha} \mathbf{1}_{\{\tau_b < \infty\}}\right] \\ &= \lim_{\alpha \downarrow 0} \frac{\mathrm{P}(\tau_b < \infty) - \mathrm{E}\,\mathrm{e}^{-\alpha \tau_b}}{\alpha} \\ &= -\lim_{\alpha \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathrm{E}\,\mathrm{e}^{-\alpha \tau_b}, \end{aligned}$$

where the last equality follows from L'Hôpital's rule. However, it follows from the implicit function theorem that

$$\lim_{\alpha \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \beta_i = \lim_{\alpha \to 0} \frac{1}{G'(\beta_i)} = \frac{1}{G'(\beta_i^*)}.$$

For $\bar{u} \geq 0$, we have $\mathrm{P}(\tau_b < \infty) = 1$ and $\mathrm{E}[\tau_b \mathbf{1}_{\{\tau_b < \infty\}}] = \mathrm{E}[\tau_b]$. Moreover, in this case, we have $\beta_1 \to 0$, $\beta_2 \to \beta_2^*$ as $\alpha \to 0$, and $G'(0) = \bar{u}$, according to Lemma 2.1.

For $\bar{u} < 0$, it is trivial that $\mathrm{E}[\tau_b] = \infty$. Moreover, in this case, $\beta_1 \to \beta_1^*$, $\beta_2 \to \beta_2^*$ as $\alpha \to 0$, where $\beta_1^*$ and $\beta_2^*$ are both positive.

The rest of the proof is a straightforward calculation, and is thus omitted.

**Corollary 3.3.** *For any $\alpha > 0$ and $\theta < \eta_1$,*

$$\mathrm{E}[\mathrm{e}^{-\alpha\tau_b+\theta X_{\tau_b}}\,\mathbf{1}_{\{\tau_b<\infty\}}] = \mathrm{e}^{\theta b}[c_1\mathrm{e}^{-b\beta_{1,\alpha}} + c_2\mathrm{e}^{-b\beta_{2,\alpha}}],$$

*where*

$$c_1 = \frac{\eta_1-\beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}\frac{\beta_{2,\alpha}-\theta}{\eta_1-\theta}, \qquad c_2 = \frac{\beta_{2,\alpha}-\eta_1}{\beta_{2,\alpha}-\beta_{1,\alpha}}\frac{\beta_{1,\alpha}-\theta}{\eta_1-\theta}.$$

*Proof.* It follows that

$$\begin{aligned}
&\mathrm{E}[\mathrm{e}^{-\alpha\tau_b+\theta X_{\tau_b}}\,\mathbf{1}_{\{\tau_b<\infty\}}]\\
&\quad= \mathrm{E}[\mathrm{e}^{-\alpha\tau_b+\theta X_{\tau_b}}\,\mathbf{1}_{\{X_{\tau_b}=b,\tau_b<\infty\}}] + \mathrm{e}^{\theta b}\,\mathrm{E}[\mathrm{e}^{-\alpha\tau_b+\theta(X_{\tau_b}-b)}\,\mathbf{1}_{\{X_{\tau_b}>b,\tau_b<\infty\}}]\\
&\quad= \mathrm{e}^{\theta b}\,\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}=b,\tau_b<\infty\}}] + \mathrm{e}^{\theta b}\frac{\eta_1}{\eta_1-\theta}\,\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}>b,\tau_b<\infty\}}]\\
&\quad= \mathrm{e}^{\theta b}\,\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}=b\}}] + \mathrm{e}^{\theta b}\frac{\eta_1}{\eta_1-\theta}\,\mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}>b\}}],
\end{aligned}$$

where we have used the conditional memoryless property. The claim follows from Theorem 3.1.

Note that, if $\bar{u} \geq 0$, then $\mathrm{P}(\tau_b < \infty) = 1$ and Corollary 3.3 implies that

$$\mathrm{E}[\mathrm{e}^{-\alpha\tau_b+\beta_{1,\alpha}X_{\tau_b}}] = 1,$$

which can be verified alternatively by applying the optional sampling theorem to the exponential martingale

$$\mathrm{e}^{\beta_{1,\alpha}X_t-G(\beta_{1,\alpha})t} = \mathrm{e}^{\beta_{1,\alpha}X_t-\alpha t}, \qquad t \geq 0.$$

**Remark 3.3.** For general Lévy processes $X$, under appropriate conditions some representations for the joint distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ can be obtained in terms of the Lévy characteristics, where $D$ is a general set and $\tau_D$ is the first passage time to $D^{\mathrm{c}}$; see [14], [28], [22] for more details.

### 3.3. Connection with renewal-type integral equations

We have used martingale and differential equations to derive closed-form solutions of the Laplace transforms for the first-passage-time probabilities. Another possible and popular approach to solving the problems, now investigated in this section, is to set up some integral equations by using renewal arguments. For simplicity, we shall only consider the case where overall drift is nonnegative, i.e. $\bar{u} \geq 0$, in which $\tau_b < \infty$ almost surely.

For any $x > 0$, define $P(x)$ as the probability that no overshoot occurs for the first passage time $\tau_x$ with $X_0 \equiv 0$, that is,

$$P(x) := \mathrm{P}(X_{\tau_x} = x).$$

**Proposition 3.1.** *The function $P(x)$ satisfies the following renewal-type integral equation:*

$$P(x+y) = P(y)P(x) + (1-P(x))\int_0^y P(y-z)\eta_1\mathrm{e}^{-\eta_1 z}\,\mathrm{d}z.$$

*However, the solution to this renewal equation is not unique. Indeed, for every $\xi \geq 0$, the function*

$$P_\xi(x) = \frac{\eta_1}{\eta_1+\xi} + \frac{\xi}{\eta_1+\xi}\mathrm{e}^{-(\eta_1+\xi)x}$$

*satisfies the integral equation with the boundary condition $P_\xi(0) = 1$.*

*Proof.* We have

$$\begin{aligned} P(x+y) &= \mathrm{P}(X_{\tau_{x+y}} = x+y) \\ &= \int_{[x,\infty)} \mathrm{P}(X_{\tau_{x+y}} = x+y \mid X_{\tau_x} \in \mathrm{d}z)\,\mathrm{P}(X_{\tau_x} \in \mathrm{d}z). \end{aligned}$$

However, Proposition 2.1 asserts that

$$\mathrm{P}(X_{\tau_x} \in \mathrm{d}z) = P(x)\delta_x(z) + (1-P(x))\eta_1 \mathrm{e}^{-\eta_1(z-x)}\,\mathrm{d}z, \qquad z \ge x;$$

here $\delta_x(\cdot)$ stands for the Dirac measure at point $\{x\}$. Therefore,

$$\begin{aligned} P(x+y) &= P(x)\,\mathrm{P}(X_{\tau_{x+y}} = x+y \mid X_{\tau_x} = x) \\ &\quad + (1-P(x)) \int_x^\infty \mathrm{P}(X_{\tau_{x+y}} = x+y \mid X_{\tau_x} \in \mathrm{d}z)\eta_1 \mathrm{e}^{-\eta_1(z-x)}\,\mathrm{d}z \\ &= P(x)P(y) + (1-P(x)) \int_x^{x+y} P(x+y-z)\eta_1 \mathrm{e}^{-\eta_1(z-x)}\,\mathrm{d}z \\ &= P(x)P(y) + (1-P(x)) \int_0^y P(y-z)\eta_1 \mathrm{e}^{-\eta_1 z}\,\mathrm{d}z, \end{aligned}$$

thanks to the strong Markov property and the fact that $\tau_b$ is finite almost surely. Now it remains to check that $P_\xi(x)$ satisfies the integral equation for every $\xi \ge 0$. To this end, note that

$$\int_0^y P_\xi(y-z)\eta_1 \mathrm{e}^{-\eta_1 z}\,\mathrm{d}z = \frac{\eta_1}{\eta_1+\xi}(1-\mathrm{e}^{-\eta_1 y}) + \frac{\eta_1}{\eta_1+\xi}\mathrm{e}^{-\eta_1 y} - \frac{\eta_1}{\eta_1+\xi}\mathrm{e}^{-(\eta_1+\xi)y}.$$

It is then very easy to check that

$$\begin{aligned} &(1-P_\xi(x)) \int_0^y P_\xi(y-z)\eta_1 \mathrm{e}^{-\eta_1 z}\,\mathrm{d}z \\ &\quad = \frac{\xi\eta_1}{(\eta_1+\xi)^2}[1-\mathrm{e}^{-(\eta_1+\xi)x} - \mathrm{e}^{-(\eta_1+\xi)y} + \mathrm{e}^{-(\eta_1+\xi)(x+y)}] \end{aligned}$$

and

$$P_\xi(x)P_\xi(y) = \frac{1}{(\eta_1+\xi)^2}\{\eta_1^2 + \eta_1\xi\mathrm{e}^{-(\eta_1+\xi)x} + \eta_1\xi\mathrm{e}^{-(\eta_1+\xi)y} + \xi^2\mathrm{e}^{-(\eta_1+\xi)(x+y)}\}.$$

Thus,

$$\begin{aligned} P_\xi(x)P_\xi(y) + (1-P_\xi(x)) \int_0^y P_\xi(y-z)\eta_1 \mathrm{e}^{-\eta_1 z}\,\mathrm{d}z &= \frac{\eta_1}{\eta_1+\xi} + \frac{\xi}{\eta_1+\xi}\mathrm{e}^{-(\eta_1+\xi)(x+y)} \\ &= P_\xi(x+y), \end{aligned}$$

and the proof is complete.

**Remark 3.4.** Proposition 3.1 shows that, in the presence of two-sided jumps, the renewal-type integral equations may not have unique solutions, mainly because of the difficulty of determining enough boundary conditions based on renewal arguments alone. It is easy to see that $\xi = -P_\xi'(0)$. Indeed, as we have shown in Corollary 3.1, it is possible to use the infinitesimal generator and differential equations to determine $\xi$. The point here is, however, that the renewal-type integral equations cannot do the job by themselves.

## 4. Joint distribution of jump diffusion and its running maxima

The probability

$$\mathrm{P}\Big(X_t \geq a, \max_{0\leq s\leq t} X_s \geq b\Big) = \mathrm{P}(X_t \geq a, \tau_b \leq t),$$

for some fixed numbers $a \leq b$ and $b > 0$, is useful, for example, in pricing barrier options while the logarithm of the underlying asset price is modelled by a jump diffusion process. In this section, we evaluate the Laplace transform $\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a, \tau_b \leq t)\,\mathrm{d}t$ for all $\alpha > 0$. It turns out that the above Laplace transform has an explicit expression, in terms of Hh functions. We shall first give a brief account of the Hh functions.

### 4.1. Hh functions

The Hh functions are defined as

$$\begin{aligned}\mathrm{Hh}_n(x) &:= \int_x^\infty \mathrm{Hh}_{n-1}(y)\,\mathrm{d}y \\ &= \frac{1}{n!}\int_x^\infty (t-x)^n \mathrm{e}^{-t^2/2}\,\mathrm{d}t, \qquad n = 0, 1, 2, \dots; \qquad (4.1)\\ \mathrm{Hh}_{-1}(x) &:= \mathrm{e}^{-x^2/2}, \\ \mathrm{Hh}_0(x) &:= \sqrt{2\pi}\,\Phi(-x),\end{aligned}$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal density. The Hh functions are nonincreasing, and have a three-term recursion, which is very useful in numerical calculation:

$$\mathrm{Hh}_n(x) = \frac{1}{n}\mathrm{Hh}_{n-2}(x) - \frac{x}{n}\mathrm{Hh}_{n-1}(x), \qquad n \geq 1; \qquad (4.2)$$

for more details, see [2, p. 691].

Introduce the following function:

$$H_i(a, b, c; n) := \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{((1/2)c^2-b)t} t^{n+i/2}\mathrm{Hh}_i\Big(c\sqrt{t} + \frac{a}{\sqrt{t}}\Big)\,\mathrm{d}t. \qquad (4.3)$$

Here $i \geq -1$, $n \geq 0$ are both integers and we make the following assumption.

**Assumption 4.1.** *The parameters $a, b, c$ are arbitrary constants such that $b > 0$ and $c > -\sqrt{2b}$.*

For $i \geq 1$, it follows from (4.2) that

$$H_i(a, b, c; n) = \frac{1}{i}H_{i-2}(a, b, c; n+1) - \frac{c}{i}H_{i-1}(a, b, c; n+1) - \frac{a}{i}H_{i-1}(a, b, c; n).$$

This recursive formula can be used to determine all the values of the $H_i$, starting from $H_{-1}(a, b, c; n)$ and $H_0(a, b, c; n)$. See Appendix A for details.

### 4.2. Laplace transform

**Proposition 4.1.** *The Laplace transform of the joint distribution is given by*

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a, \tau_b \leq t)\,\mathrm{d}t\\ &\qquad = A\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a-b)\,\mathrm{d}t + B\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t+\xi^+ \geq a-b)\,\mathrm{d}t.\end{aligned}$$

*Here* $\xi^+$ *is an independent exponential random variable with rate* $\eta_1 > 0$ *and*

$$A := \mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}=b\}}] = \frac{\eta_1-\beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}\mathrm{e}^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha}-\eta_1}{\beta_{2,\alpha}-\beta_{1,\alpha}}\mathrm{e}^{-b\beta_{2,\alpha}}, \tag{4.4}$$

$$B := \mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}>b\}}] = \frac{(\eta_1-\beta_{1,\alpha})(\beta_{2,\alpha}-\eta_1)}{\eta_1(\beta_{2,\alpha}-\beta_{1,\alpha})}[\mathrm{e}^{-b\beta_{1,\alpha}} - \mathrm{e}^{-b\beta_{2,\alpha}}]. \tag{4.5}$$

*Proof.* We need to compute two integrals:

$$I_1 = \int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a, X_{\tau_b} = b, \tau_b \leq t)\,\mathrm{d}t,$$
$$I_2 = \int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a, X_{\tau_b} > b, \tau_b \leq t)\,\mathrm{d}t.$$

For the first one,

$$\begin{aligned}I_1 &= \int_0^\infty \mathrm{e}^{-\alpha t}\int_0^t \mathrm{P}(X_t \geq a, X_{\tau_b} = b, \tau_b \in \mathrm{d}s)\,\mathrm{d}t\\ &= \int_0^\infty\int_0^t \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_{\tau_b} = b, \tau_b \in \mathrm{d}s)\,\mathrm{P}(X_{t-s} \geq a-b)\,\mathrm{d}t\\ &= \int_0^\infty \mathrm{e}^{-\alpha s}\,\mathrm{P}(X_{\tau_b} = b, \tau_b \in \mathrm{d}s)\int_0^\infty \mathrm{e}^{-\alpha u}\,\mathrm{P}(X_u \geq a-b)\,\mathrm{d}u\\ &= \mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}=b\}}]\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a-b)\,\mathrm{d}t,\end{aligned}$$

where the second equality follows from the strong Markov property, and the third equality follows from the fact that the Laplace transform of a convolution is the product of Laplace transforms.

As for the second integral, observe that, for any $s \in [0, t]$,

$$\mathrm{P}(X_t \geq a, X_{\tau_b} > b, \tau_b \in \mathrm{d}s) = \mathrm{P}(X_{\tau_b} > b, \tau_b \in \mathrm{d}s)\,\mathrm{P}(X_{t-s} + \xi^+ \geq a-b),$$

by the conditional memoryless property and the conditional independence (see Proposition 2.1), as well as the strong Markov property; here $\xi^+$ is some independent exponential random variable with rate $\eta_1$. It follows exactly as for $I_1$ that

$$I_2 = \mathrm{E}[\mathrm{e}^{-\alpha\tau_b}\,\mathbf{1}_{\{X_{\tau_b}>b\}}]\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t + \xi^+ \geq a-b)\,\mathrm{d}t,$$

from which the proof is completed.

**Theorem 4.1.** *The Laplace transform of the joint distribution can be further written as*

$$\begin{aligned}\int_0^\infty & \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \ge a, \tau_b \le t)\,\mathrm{d}t \\ &= (A+B)\sum_{n=0}^\infty \frac{\lambda^n}{n!} H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) \\ &\quad + \mathrm{e}^{h\sigma\eta_1}\sum_{n=1}^\infty\sum_{j=1}^n \frac{\lambda^n}{n!}(AP_{n,j} + B\bar{P}_{n,j})\Big(\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n)\Big) \\ &\quad - \mathrm{e}^{-h\sigma\eta_2}\sum_{n=1}^\infty\sum_{j=1}^n \frac{\lambda^n}{n!}(AQ_{n,j} + B\bar{Q}_{n,j})\Big(\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h, \Upsilon_\alpha, c_-; n)\Big) \\ &\quad + \mathrm{e}^{h\sigma\eta_1} B\sum_{n=1}^\infty\sum_{i=0}^n \frac{(\lambda p)^n}{n!}(\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n) + \mathrm{e}^{h\sigma\eta_1} BH_0(h, \Upsilon_\alpha, c_+; 0).\end{aligned}$$

*Here*

$$P_{n,i} := \sum_{j=i}^{n-1}\binom{n}{j} p^j q^{n-j}\binom{n-i-1}{j-i}\Big(\frac{\eta_1}{\eta_1+\eta_2}\Big)^{j-i}\Big(\frac{\eta_2}{\eta_1+\eta_2}\Big)^{n-j}, \qquad 1 \le i \le n-1,$$

$$Q_{n,i} := \sum_{j=i}^{n-1}\binom{n}{j} q^j p^{n-j}\binom{n-i-1}{j-i}\Big(\frac{\eta_2}{\eta_1+\eta_2}\Big)^{j-i}\Big(\frac{\eta_1}{\eta_1+\eta_2}\Big)^{n-j}, \qquad 1 \le i \le n-1,$$

*while* $P_{n,n} := p^n$ *and* $Q_{n,n} := q^n$*;*

$$\bar{P}_{n,1} := \sum_{i=1}^n Q_{n,i}\Big(\frac{\eta_2}{\eta_1+\eta_2}\Big)^i, \qquad \bar{P}_{n,i} = P_{n,i-1}, \quad 2 \le i \le n+1,$$

$$\bar{Q}_{n,i} := \sum_{j=i}^{n}\binom{n}{j} q^j p^{n-j}\binom{n-i}{j-i}\Big(\frac{\eta_2}{\eta_1+\eta_2}\Big)^{j-i}\Big(\frac{\eta_1}{\eta_1+\eta_2}\Big)^{n-j+1}, \qquad 1 \le i \le n;$$

$$c_+ := \sigma\eta_1 + \frac{\mu}{\sigma}, \qquad c_- := \sigma\eta_2 - \frac{\mu}{\sigma}, \qquad \Upsilon_\alpha := \alpha + \lambda + \frac{\mu^2}{2\sigma^2}, \qquad h := \frac{b-a}{\sigma}, \tag{4.6}$$

*and A and B are given by (4.4) and (4.5).*

The proof of this theorem is long and is given in Appendix B.

**Remark 4.1.** All the parameters involved in the functions $H_i$ in Theorem 4.1 satisfy Assumption 4.1.

**Remark 4.2.** It is easy to derive the corresponding result for $\mathrm{P}(X_t \le -a,\ \tilde{\tau}_{-b} \le t)$, $a \le b$, $b > 0$, where $\tilde{\tau}_{-b} := \inf\{t \ge 0 :\ X_t \le -b\}$. More precisely, we only need to make the following changes in Theorem 4.1: $p \mapsto q$, $q \mapsto p$, $\beta_{1,\alpha} \mapsto \beta_{3,\alpha}$, $\eta_1 \mapsto \eta_2$, $\eta_2 \mapsto \eta_1$, and $\beta_{2,\alpha} \mapsto \beta_{4,\alpha}$.

## 5. Laplace inversion and numerical examples

Since the distributions of the first passage times are given in terms of Laplace transforms, numerical inversion of Laplace transforms becomes necessary. To do this, we shall use the Gaver–Stehfest algorithm. The reason is that, among all the Laplace inversion algorithms, to the best of the authors' knowledge, the Gaver–Stehfest is the only one that does the inversion on the real line; all others perform the calculation in the complex domain, and so are unsuitable for our purpose as the Laplace transforms in our case involve finding the roots $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$. See [1] for a survey of Laplace inversion algorithms.

The algorithm is very simple. For any bounded real-valued function $f(\cdot)$ defined on $[0, \infty)$ that is continuous at $t$,

$$f(t) = \lim_{n\to\infty} \tilde{f}_n(t),$$

where

$$\tilde{f}_n(t) = \frac{\ln(2)}{t} \frac{(2n)!}{n!\,(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \hat{f}\left((n+k)\frac{\ln(2)}{t}\right) \tag{5.1}$$

and $\hat{f}$ is the Laplace transform of $f$, i.e. $\hat{f}(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha t} f(t)\,\mathrm{d}t$. To speed up the convergence, an $n$-point Richardson extrapolation can be used. More precisely, $f(t)$ can be approximated by $f_n^*(t)$ for large $n$, where

$$f_n^*(t) = \sum_{k=1}^{n} w(k, n)\tilde{f}_k(t),$$

and the extrapolation weights $w(k, n)$ are given by

$$w(k, n) = (-1)^{n-k} \frac{k^n}{k!\,(n-k)!}. \tag{5.2}$$

Numerically, we find that it is better to ignore the first few initial calculations of $\tilde{f}_k$. As a result, the algorithm approximates $f(t)$ by $f_n^*(t)$, where

$$f_n^*(t) = \sum_{k=1}^{n} w(k, n)\tilde{f}_{k+B}(t),$$

with $\tilde{f}$ and $w$ given by (5.1) and (5.2), and $B \geq 0$ is the initial burning-out number (typically equal to 2 or 3).

The main advantages of the Gaver–Stehfest algorithm are: (a) it is very easy to program (several lines of code will do the job); (b) it converges very quickly; as we will see, the algorithm typically converges nicely even for $n$ between 5 and 10; (c) it is stable (i.e. a small perturbation of initial inputs will not lead to a dramatic change of final results) if high-accuracy computation is used. The main disadvantage of the algorithm is that it needs high accuracy, as both $\tilde{f}_n$ and the weights $w(k, n)$ involve factorials and alternative $+/-$ signs. The interested reader may refer to [1] for more discussions on the algorithm. In our numerical examples, an accuracy of 30–80 digits is typically needed. However, in many software packages (e.g. MATHEMATICA®) an arbitrary accuracy can be specified, and in standard programming languages (e.g. C++) subroutines for high-precision calculation are available. So this is not a big problem.

It is easy to compute the marginal and joint distributions of the first passage times for the double exponential jump diffusion process by using the Laplace transform formulae given in

TABLE 1: The cases of positive overall drifts $\bar{u} > 0$ ($\mu = 0.1$). The Monte Carlo results are based on 16 000 simulation runs.

| | $P(\tau_b \le t)$ | | $P(\tau_b \le t,\ X_t \ge a)$ | |
|---|---|---|---|---|
| $n$ | $\lambda = 0.01$ | $\lambda = 3$ | $\lambda = 0.01$ | $\lambda = 3$ |
| 1 | 0.34669 | 0.33472 | 0.30266 | 0.28114 |
| 2 | 0.30818 | 0.29912 | 0.27062 | 0.25673 |
| 3 | 0.28211 | 0.27521 | 0.24940 | 0.23849 |
| 4 | 0.26880 | 0.26313 | 0.23886 | 0.22886 |
| 5 | 0.26328 | 0.25819 | 0.23464 | 0.22507 |
| 6 | 0.26136 | 0.25649 | 0.23324 | 0.22393 |
| 7 | 0.26078 | 0.25599 | 0.23285 | 0.22367 |
| 8 | 0.26063 | 0.25587 | 0.23277 | 0.22363 |
| 9 | 0.26060 | 0.25585 | 0.23275 | 0.22362 |
| 10 | 0.26060 | 0.25584 | 0.23275 | 0.22362 |
| Total CPU time | 1.26 sec | 1.76 sec | 4.53 min | 4.61 min |
| Brownian motion case | 0.26061 | N.A. | 0.23278 | N.A. |
| Monte Carlo simulation | | | | |
| 200 points<br>CPU time: 15 min<br>Point est. and 95% C.I. | 0.248<br>(0.241, 0.255) | 0.244<br>(0.236, 0.252) | 0.226<br>(0.220, 0.232) | 0.218<br>(0.211, 0.225) |
| 2000 points<br>CPU time: 1 hr 20 min<br>Point est. and 95% C.I. | 0.254<br>(0.247, 0.261) | 0.251<br>(0.244, 0.258) | 0.227<br>(0.220, 0.234) | 0.220<br>(0.214, 0.226) |

Sections 3 and 4, in conjunction with the Gaver–Stehfest algorithm. As a numerical illustration, we shall present two examples; one is to compute $P(\tau_b \le t)$ and the other $P(\tau_b \le t,\ X_t \ge a)$ for $b = 0.3$, $a = 0.2$, and $t = 1$. The results are presented in Tables 1 and 2. The parameters, which are chosen to reflect those in typical finance applications, for the double exponential jump diffusion are $\mu = \pm 0.1$, $\sigma = 0.2$, $p = 0.5$, $\eta_1 = 1/0.02$, $\eta_2 = 1/0.03$, and $\lambda = 3$. To make a comparison with the Monte Carlo simulation, we also use $\lambda = 0.01$, so that the results may be compared with the limiting Brownian motion case ($\lambda = 0$); the formulae for the first passage times of Brownian motion can be found in many textbooks, e.g. [17].

All the computations are done on a Pentium® 400 MHz PC. The initial burning-out number used in all calculations is $B = 2$. Also, in calculating $P(\tau_b \le t,\ X_t \ge a)$, we truncate the Poisson sum after the tenth term, as additional numerical calculations suggest that the error involved in the truncation is less than $10^{-6}$. The reason why the calculation of $P(\tau_b \le t,\ X_t \ge a)$ takes a longer time is that it requires that the functions $H_i$ are computed recursively and MATHEMATICA is slow at recursive calculation.

To speed up the simulation, binomial approximation is used to simulate the Poisson processes. Note that the Monte Carlo simulation is biased and slow, due to two sources of errors: random sampling error and systematic discretization bias. It is quite possible to significantly reduce the random sampling error here (and, thus, the width of the confidence intervals) by using some variance reduction techniques, such as control variates and importance sampling (suitable for the case of $\bar{u} < 0$). The systematic discretization bias, resulting from approximating a continuous-time process by a discrete-time process in simulation, is, however, very difficult to reduce; in

TABLE 2: The cases of negative overall drifts $\bar{u} < 0$ ($\mu = -0.1$). The Monte Carlo results are based on 16 000 simulation runs.

| | $\mathrm{P}(\tau_b \le t)$ | | $\mathrm{P}(\tau_b \le t,\ X_t \ge a)$ | |
|---|---|---|---|---|
| $n$ | $\lambda = 0.01$ | $\lambda = 3$ | $\lambda = 0.01$ | $\lambda = 3$ |
| 1 | 0.07737 | 0.07884 | 0.04762 | 0.04626 |
| 2 | 0.06878 | 0.07096 | 0.04591 | 0.04558 |
| 3 | 0.06296 | 0.06562 | 0.04455 | 0.04480 |
| 4 | 0.05999 | 0.06289 | 0.04376 | 0.04428 |
| 5 | 0.05876 | 0.06176 | 0.04340 | 0.04404 |
| 6 | 0.05833 | 0.06137 | 0.04328 | 0.04397 |
| 7 | 0.05820 | 0.06126 | 0.04325 | 0.04396 |
| 8 | 0.05817 | 0.06123 | 0.04325 | 0.04396 |
| 9 | 0.05816 | 0.06122 | 0.04325 | 0.04397 |
| 10 | 0.05816 | 0.06122 | 0.04325 | 0.04397 |
| Total CPU time | 1.20 sec | 1.81 sec | 4.49 min | 4.67 min |
| Brownian motion case | 0.05815 | N.A. | 0.04324 | N.A. |
| Monte Carlo simulation | | | | |
| 200 points<br>CPU time: 15 min<br>Point est. and 95% C.I. | 0.055<br>(0.051, 0.059) | 0.056<br>(0.052, 0.060) | 0.042<br>(0.038, 0.046) | 0.043<br>(0.040, 0.046) |
| 2000 points<br>CPU time: 1 hr 20 min<br>Point est. and 95% C.I. | 0.057<br>(0.053, 0.061) | 0.059<br>(0.055, 0.063) | 0.043<br>(0.040, 0.046) | 0.044<br>(0.041, 0.047) |

the examples given above, it makes the calculation from the simulation biased low. Even in the Brownian motion case, because of the presence of a boundary, the discretization bias is significant, resulting in a surprisingly slow rate of convergence for simulating the first passage time, both theoretically and numerically; e.g. in [3] it is shown that the discretization error has an order $\frac{1}{2}$, which is much slower than the order-1 convergence for simulation without the boundary; 16 000 points are suggested for a Brownian motion with $\mu = -1$, $\sigma = 1$, and time $t = 8$. In the presence of jumps, the discretization bias is even more serious, especially for large $t$ or $\lambda$. This explains the large bias in our simulation results.

## Appendix A. Computation of $H_i$

We defined in (4.3) the following function:

$$H_i(a, b, c; n) := \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{((1/2)c^2-b)t} t^{n+i/2} \mathrm{Hh}_i\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) \mathrm{d}t$$

for integers $i \ge -1$, $n \ge 0$. We assume that Assumption 4.1 holds throughout this section, that is, $b > 0$ and $c > -\sqrt{2b}$. The following recursion formula holds:

$$H_i(a, b, c; n) = \frac{1}{i} H_{i-2}(a, b, c; n+1) - \frac{c}{i} H_{i-1}(a, b, c; n+1) - \frac{a}{i} H_{i-1}(a, b, c; n).$$

Therefore, it suffices to evaluate $H_{-1}(a, b, c; n)$ and $H_0(a, b, c; n)$, both of which can be calculated explicitly.

**Lemma A.1.** *If $a \neq 0$, then, for all integers $n \geq 0$,*

$$H_{-1}(a,b,c;n) = \mathrm{e}^{-ac-\sqrt{2a^2b}}\sqrt{\frac{1}{2b}}\left(\sqrt{\frac{a^2}{2b}}\right)^n \sum_{j=0}^{n} \frac{(-n)_j(n+1)_j}{j!\,(-2\sqrt{2a^2b}\,)^j},$$

*and, for all integers $n \leq -1$,*

$$H_{-1}(a,b,c;n) = \mathrm{e}^{-ac-\sqrt{2a^2b}}\sqrt{\frac{1}{2b}}\left(\sqrt{\frac{a^2}{2b}}\right)^n \sum_{j=0}^{-n-1} \frac{(-n)_j(n+1)_j}{j!\,(-2\sqrt{2a^2b}\,)^j},$$

*where $(n)_j := n(n+1)\cdots(n+j-1)$ for all integers $n$ with the convention that $(n)_0 \equiv 1$. If $a = 0$, then, for all integers $n \geq 0$,*

$$H_{-1}(0,b,c;n) = \frac{(2n)!}{n!\,(4b)^n}\sqrt{\frac{1}{2b}},$$

*and, for all integers $n \leq -1$,*

$$H_{-1}(0,b,c;n) = +\infty.$$

*Proof.* We shall prove the case of $a \neq 0$ first. Since $\mathrm{Hh}_{-1}(x) = \mathrm{e}^{-x^2/2}$, by definition

$$\begin{aligned} H_{-1}(a,b,c;n) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{((1/2)c^2-b)t}t^{n-1/2}\mathrm{e}^{-(1/2)(c\sqrt{t}+a/\sqrt{t})^2}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ac}\int_0^\infty \mathrm{e}^{-(bt+a^2/2t)}t^{n-1/2}\,\mathrm{d}t. \end{aligned}$$

Recall the so-called modified Bessel function of the third kind [4, p. 5], $K_\nu(x)$, which has an integral representation [5, p. 146] as

$$\frac{\alpha^\nu}{2}\int_0^\infty \mathrm{e}^{-(z/2)(t+\alpha^2/t)}\frac{1}{t^{\nu+1}}\,\mathrm{d}t = K_\nu(\alpha z)$$

for arbitrary constants $\nu$ and $\alpha > 0$, $z > 0$. It is easy to show that

$$H_{-1}(a,b,c;n) = \sqrt{\frac{2}{\pi}}\mathrm{e}^{-ac}\left(\sqrt{\frac{a^2}{2b}}\right)^{n+1/2} K_{-(n+1/2)}(\sqrt{2a^2b}\,).$$

However, the modified Bessel function $K_\nu(x)$ has the property that

$$K_\nu(x) = K_{-\nu}(x) \quad \text{for all } \nu,$$

$$K_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\sum_{j=0}^{n}\frac{(-n)_j(n+1)_j}{j!\,(-2x)^j} \quad \text{for all } n \geq 0;$$

see [4, pp. 5, 10]. The result follows.

Now consider the case of $a = 0$. By definition,

$$H_{-1}(0, b, c; n) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-bt} t^{n-1/2}\,\mathrm{d}t.$$

For $n \le -1$, the integral is obviously $+\infty$. For $n \ge 0$, this integral is the Laplace transform of $t^{n-1/2}$, which can be found in many integral tables, and we have

$$H_{-1}(0, b, c; n) = \frac{1}{\sqrt{2\pi}} \frac{(2n)!}{n!\,(4b)^n} \sqrt{\frac{\pi}{b}} = \frac{(2n)!}{n!\,(4b)^n} \sqrt{\frac{1}{2b}}.$$

This completes the proof.

The following lemma gives the value of $H_0$ in terms of $H_{-1}$.

**Lemma A.2.** *Suppose that $n \ge 0$ is an integer.*

1. *If $b = \frac{1}{2}c^2$, then*

$$H_0(a, b, c; n) = \frac{c}{2(n+1)} H_{-1}(a, b, c; n+1) - \frac{a}{2(n+1)} H_{-1}(a, b, c; n).$$

2. *If $b \ne \frac{1}{2}c^2$, $a > 0$, then*

$$H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} \times \sum_{i=0}^{n} \frac{(b - \frac{1}{2}c^2)^i}{i!} \left( \frac{a}{2} H_{-1}(a, b, c; i-1) - \frac{c}{2} H_{-1}(a, b, c; i) \right).$$

3. *If $b \ne \frac{1}{2}c^2$, $a < 0$, then*

$$H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} + \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} \times \sum_{i=0}^{n} \frac{(b - \frac{1}{2}c^2)^i}{i!} \left( \frac{a}{2} H_{-1}(a, b, c; i-1) - \frac{c}{2} H_{-1}(a, b, c; i) \right).$$

4. *If $b \ne \frac{1}{2}c^2$, $a = 0$, then*

$$H_0(0, b, c; n) = \frac{n!}{2(b - \frac{1}{2}c^2)^{n+1}} - \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} \sum_{i=0}^{n} \frac{(b - \frac{1}{2}c^2)^i}{i!} \frac{c}{2} H_{-1}(0, b, c; i).$$

*Proof.* It follows from the definition (4.1) of Hh that

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{Hh}_n(x) = -\mathrm{Hh}_{n-1}(x), \qquad n = 0, 1, 2, \ldots.$$

For $b = \frac{1}{2}c^2$ (i.e. $c = \sqrt{2b} > -\sqrt{2b}$), since $\mathrm{Hh}_0(x) = \sqrt{2\pi}\,\Phi(-x)$, we have

$$\begin{aligned}
H_0(a,b,c;n) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty t^n \mathrm{Hh}_0\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \frac{t^{n+1}}{n+1} \mathrm{Hh}_0\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right)\Big|_0^\infty \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{t^{n+1}}{n+1} \mathrm{Hh}_{-1}\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right)\left(\frac{c}{2\sqrt{t}} - \frac{a}{2t\sqrt{t}}\right) \mathrm{d}t \\
&= \frac{c}{2(n+1)} \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{n+1/2} \mathrm{Hh}_{-1}\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) \mathrm{d}t \\
&\quad - \frac{a}{2(n+1)} \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{n-1/2} \mathrm{Hh}_{-1}\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) \mathrm{d}t \\
&= \frac{c}{2(n+1)} H_{-1}(a,b,c;n+1) - \frac{a}{2(n+1)} H_{-1}(a,b,c;n).
\end{aligned}$$

For $b \neq \frac{1}{2}c^2$, we have the following elementary identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-n!}{(b-\frac{1}{2}c^2)^{n+1}} \mathrm{e}^{((1/2)c^2-b)t} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i t^i}{i!}\right) = \mathrm{e}^{((1/2)c^2-b)t} t^n,$$

and (noting that $c > -\sqrt{2b}$)

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\left(\frac{-n!}{(b-\frac{1}{2}c^2)^{n+1}} \mathrm{e}^{((1/2)c^2-b)t} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i t^i}{i!}\right) \mathrm{Hh}_0\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right)\Big|_0^\infty \\
&\quad = \begin{cases} 0 & \text{if } a > 0, \\ \dfrac{n!}{(b-\frac{1}{2}c^2)^{n+1}} & \text{if } a < 0, \\ \dfrac{n!}{2(b-\frac{1}{2}c^2)^{n+1}} & \text{if } a = 0. \end{cases}
\end{aligned}$$

The rest of the proof is simply integration by parts, and is thus omitted.

## Appendix B. Proof of Theorem 4.1

The proof relies on four lemmas, of which the first two are Propositions B.2 and B.3 in [18]. The third can be proved by a modification of Proposition B.1 in [18]. Thus, we only give a proof for the last lemma.

**Lemma B.1.** *Suppose that* $\{\xi_1, \xi_2, \dots\}$ *is a sequence of i.i.d. exponential random variables with rate* $\eta$*,* $Z$ *is a normal random variable with distribution* $N(0, \sigma^2)$*, and the* $\xi_j$ *and* $Z$ *are independent. For* $n \geq 1$*,*

1. *the tail probability of the random variable* $Z + \sum_{i=1}^{n} \xi_i$ *is given by*

$$\mathrm{P}\left(Z + \sum_{i=1}^{n} \xi_i \geq x\right) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} \mathrm{e}^{(1/2)(\sigma\eta)^2} I_{n-1}\left(x; -\eta, -\frac{1}{\sigma}, -\sigma\eta\right);$$

2. *the tail probability of the random variable* $Z - \sum_{i=1}^{n} \xi_i$ *is given by*

$$\mathrm{P}\bigg(Z - \sum_{i=1}^{n} \xi_i \geq x\bigg) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} \mathrm{e}^{(1/2)(\sigma\eta)^2} I_{n-1}\bigg(x; \eta, \frac{1}{\sigma}, -\sigma\eta\bigg).$$

*Here the function* $I_n$ *is defined as*

$$I_n(c; \alpha, \beta, \delta) := \int_c^{\infty} \mathrm{e}^{\alpha x} \mathrm{Hh}_n(\beta x - \delta)\, \mathrm{d}x.$$

**Lemma B.2.** *If* $\beta > 0$, $\alpha \neq 0$, *then*

$$\begin{aligned} I_n(c; \alpha, \beta, \delta) = &-\frac{\mathrm{e}^{\alpha c}}{\alpha} \sum_{i=0}^{n} \bigg(\frac{\beta}{\alpha}\bigg)^{n-i} \mathrm{Hh}_i(\beta c - \delta) \\ &+ \bigg(\frac{\beta}{\alpha}\bigg)^{n+1} \frac{\sqrt{2\pi}}{\beta} \exp\bigg(\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\bigg) \Phi\bigg(-\beta c + \delta + \frac{\alpha}{\beta}\bigg). \end{aligned}$$

*If* $\beta < 0$, $\alpha < 0$, *then*

$$\begin{aligned} I_n(c; \alpha, \beta, \delta) = &-\frac{\mathrm{e}^{\alpha c}}{\alpha} \sum_{i=0}^{n} \bigg(\frac{\beta}{\alpha}\bigg)^{n-i} \mathrm{Hh}_i(\beta c - \delta) \\ &- \bigg(\frac{\beta}{\alpha}\bigg)^{n+1} \frac{\sqrt{2\pi}}{\beta} \exp\bigg(\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\bigg) \Phi\bigg(\beta c - \delta - \frac{\alpha}{\beta}\bigg). \end{aligned}$$

*If* $\beta > 0$, $\alpha = 0$, *then*

$$\begin{aligned} I_n(c; 0, \beta, \delta) &= \int_c^{\infty} \mathrm{Hh}_n(\beta x - \delta)\, \mathrm{d}x \\ &= \frac{1}{\beta} \mathrm{Hh}_{n+1}(\beta c - \delta). \end{aligned}$$

**Lemma B.3.** *For any fixed* $t \geq 0$, *conditioning on* $N_t = n$, $n \geq 1$, $X_t$ *has a decomposition in distribution*

$$X_t \sim \begin{cases} \mu t + Z + \sum_{i=1}^{j} \xi_i^{+} \text{ with probability } P_{n,j}, & j = 1, 2, \ldots, n, \\ \mu t + Z - \sum_{i=1}^{j} \xi_i^{-} \text{ with probability } Q_{n,j}, & j = 1, 2, \ldots, n, \end{cases}$$

*and* $X_t + \xi^{+}$ *has a decomposition in distribution*

$$X_t + \xi^{+} \sim \begin{cases} \mu t + Z + \sum_{i=1}^{j} \xi_i^{+} \text{ with probability } \bar{P}_{n,j}, & j = 1, 2, \ldots, n+1, \\ \mu t + Z - \sum_{i=1}^{j} \xi_i^{-} \text{ with probability } \bar{Q}_{n,j}, & j = 1, 2, \ldots, n, \end{cases}$$

*where* $Z$ *is a normal random variable with distribution* $N(0, \sigma^2 t)$, $\{\xi_i^{+}; i \geq 1\}$, $\{\xi_i^{-}; i \geq 1\}$ *are i.i.d. exponential random variables with rates* $\eta_1$ *and* $\eta_2$ *respectively, and* $Z$ *and the* $\xi_j^{\pm}$ *are independent.*

**Lemma B.4.** *We have*

$$\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t} t^n \frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}} \mathrm{e}^{(1/2)(\eta_1\sigma)^2 t} I_{j-1}\Big(-h\sigma - \mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big)\,\mathrm{d}t$$

$$= H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) + \mathrm{e}^{\eta_1 h\sigma} \sum_{i=0}^{j-1} (\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n), \tag{B.1}$$

$$\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t} t^n \frac{(\sigma\sqrt{t}\eta_2)^j}{\sigma\sqrt{2\pi t}} \mathrm{e}^{(1/2)(\eta_2\sigma)^2 t} I_{j-1}\Big(-h\sigma - \mu t; \eta_2, \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\Big)\,\mathrm{d}t$$

$$= H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) - \mathrm{e}^{-h\sigma\eta_2} \sum_{i=0}^{j-1} (\sigma\eta_2)^i H_i(-h, \Upsilon_\alpha, c_-; n). \tag{B.2}$$

*Proof.* In (B.1) the function $I_{j-1}$, $j \geq 1$, can be split into two summands by Lemma B.2:

$$I_{j-1} = \frac{\sigma\sqrt{2\pi t}}{(\sigma\sqrt{t}\eta_1)^j} \mathrm{e}^{-(1/2)(\eta_1\sigma)^2 t} \Phi\Big(\frac{h}{\sqrt{t}} + \frac{\mu}{\sigma}\sqrt{t}\Big)$$

$$+ \frac{\mathrm{e}^{\eta_1 h\sigma}}{\eta_1} \mathrm{e}^{\mu\eta_1 t} \sum_{i=0}^{j-1} \Big(\frac{1}{\sigma\sqrt{t}\eta_1}\Big)^{j-i-1} \mathrm{Hh}_i\Big(\frac{h}{\sqrt{t}} + c_+\sqrt{t}\Big).$$

The first summand will contribute

$$\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t} t^n \frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}} \mathrm{e}^{(1/2)(\eta_1\sigma)^2 t} \frac{\sigma\sqrt{2\pi t}}{(\sigma\sqrt{t}\eta_1)^j} \mathrm{e}^{-(1/2)(\eta_1\sigma)^2 t} \Phi\Big(\frac{h}{\sqrt{t}} + \frac{\mu}{\sigma}\sqrt{t}\Big)\,\mathrm{d}t$$

$$= \int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t} t^n \frac{1}{\sqrt{2\pi}} \mathrm{Hh}_0\Big(-\frac{h}{\sqrt{t}} - \frac{\mu}{\sigma}\sqrt{t}\Big)\,\mathrm{d}t$$

$$= H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big).$$

The contribution from the second summand is

$$\sum_{i=0}^{j-1} \int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t} t^n \frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}} \mathrm{e}^{(1/2)(\eta_1\sigma)^2 t} \frac{\mathrm{e}^{\eta_1 h\sigma}}{\eta_1} \mathrm{e}^{\mu\eta_1 t} \Big(\frac{1}{\sigma\sqrt{t}\eta_1}\Big)^{j-i-1} \mathrm{Hh}_i\Big(\frac{h}{\sqrt{t}} + c_+\sqrt{t}\Big)\,\mathrm{d}t$$

$$= \mathrm{e}^{\eta_1 h\sigma} \sum_{i=0}^{j-1} (\sigma\eta_1)^i \int_0^\infty \mathrm{e}^{(-\alpha-\lambda+\mu\eta_1+(1/2)\sigma^2\eta_1^2)t} t^{n+i/2} \frac{1}{\sqrt{2\pi}} \mathrm{Hh}_i\Big(\frac{h}{\sqrt{t}} + c_+\sqrt{t}\Big)\,\mathrm{d}t$$

$$= \mathrm{e}^{\eta_1 h\sigma} \sum_{i=0}^{j-1} (\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n).$$

The proof for (B.2) is similar, and thus omitted.

*Proof of Theorem 4.1.* From Proposition 4.1 the Laplace transform of the joint distribution is given by

$$\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a, \tau_b \leq t)\,\mathrm{d}t = A\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \geq a-b)\,\mathrm{d}t + B\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t+\xi^+ \geq a)\,\mathrm{d}t,$$

where $\xi^+$ is an independent exponential distribution with rate $\eta_1$.

Let us consider the first term. By conditioning on $N_t$, it follows from Lemmas B.1 and B.3 that

$$\begin{aligned}\mathrm{P}(X_t \geq a-b) &= \mathrm{e}^{-\lambda t}\Phi\Big(\frac{\mu t-(a-b)}{\sigma\sqrt{t}}\Big)\\ &\quad+\sum_{n=1}^\infty \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{j=1}^n P_{n,j}\frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_1\sigma)^2 t}\\ &\qquad\times I_{j-1}\Big(a-b-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big)\\ &\quad+\sum_{n=1}^\infty \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{j=1}^n Q_{n,j}\frac{(\sigma\sqrt{t}\eta_2)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_2\sigma)^2 t}\\ &\qquad\times I_{j-1}\Big(a-b-\mu t; \eta_2, \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\Big).\end{aligned}\tag{B.3}$$

Using the identity $\mathrm{Hh}_0(x)=\sqrt{2\pi}\,\Phi(-x)$ and the definition of $H$ in (4.3), it is easy to check that the first term in (B.3) will contribute

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t}\frac{1}{\sqrt{2\pi}}\mathrm{Hh}_0\Big(\frac{-\mu t+(a-b)}{\sigma\sqrt{t}}\Big)\,\mathrm{d}t &= H_0\Big(\frac{a-b}{\sigma}, \alpha+\lambda+\frac{\mu^2}{2\sigma^2}, -\frac{\mu}{\sigma}; 0\Big)\\ &= H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; 0\Big),\end{aligned}$$

in the notation of (4.6). The second term in (B.3) will contribute

$$\sum_{n=1}^\infty\sum_{j=1}^n \frac{\lambda^n}{n!}P_{n,j} \times\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t}t^n\frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_1\sigma)^2 t}I_{j-1}\Big(-h\sigma-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big)\,\mathrm{d}t$$

(note that $a-b=-h\sigma$). The third term in (B.3) will contribute

$$\sum_{n=1}^\infty\sum_{j=1}^n \frac{\lambda^n}{n!}Q_{n,j} \times\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t}t^n\frac{(\sigma\sqrt{t}\eta_2)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_2\sigma)^2 t}I_{j-1}\Big(-h\sigma-\mu t; \eta_2, \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\Big)\,\mathrm{d}t.$$

Therefore, by Lemma B.4,

$$
\begin{aligned}
\int_0^\infty & \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t \ge a-b)\,\mathrm{d}t \\
&= H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; 0\Big) \\
&\quad + \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{\lambda^n}{n!}(P_{n,j}+Q_{n,j}) H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) \\
&\quad + \mathrm{e}^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}P_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h,\Upsilon_\alpha, c_+; n) \\
&\quad - \mathrm{e}^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}Q_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h,\Upsilon_\alpha, c_-; n) \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!} H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) \\
&\quad + \mathrm{e}^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}P_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h,\Upsilon_\alpha, c_+; n) \\
&\quad - \mathrm{e}^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}Q_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h,\Upsilon_\alpha, c_-; n),
\end{aligned}
$$

where the last equality follows from the fact that $\sum_{j=1}^{n}(P_{n,j}+Q_{n,j})=1$.

It remains to evaluate the Laplace transform

$$\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t+\xi^+ \ge a-b)\,\mathrm{d}t,$$

where $\xi^+$ is an independent exponential random variable with rate $\eta_1$. Conditioning on $N_t=0$, clearly $X_t+\xi^+ \sim \mu t + Z + \xi^+$, where $Z$ is a normal random variable with distribution $N(0,\sigma^2 t)$. Therefore, by conditioning on $N_t$, we have, via Lemmas B.3 and B.1, that

$$
\begin{aligned}
\mathrm{P}(X_t+\xi^+ \ge a-b) &= \mathrm{e}^{-\lambda t} I_0\Big(a-b-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big)\frac{\eta_1}{\sqrt{2\pi}}\mathrm{e}^{(1/2)\sigma^2\eta_1^2 t} \\
&\quad + \sum_{n=1}^{\infty}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{j=1}^{n+1}\bar{P}_{n,j}\frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_1\sigma)^2 t} \\
&\qquad\qquad \times I_{j-1}\Big(a-b-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big) \\
&\quad + \sum_{n=1}^{\infty}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{j=1}^{n}\bar{Q}_{n,j}\frac{(\sigma\sqrt{t}\eta_2)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_2\sigma)^2 t} \\
&\qquad\qquad \times I_{j-1}\Big(a-b-\mu t; \eta_2, \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\Big).
\end{aligned}
\tag{B.4}
$$

The first term in (B.4) is

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t} I_0\Big(-h\sigma-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big)\frac{\eta_1}{\sqrt{2\pi}}\mathrm{e}^{(1/2)\sigma^2\eta_1^2 t}\\
&\quad= \mathrm{e}^{-\lambda t}\frac{\eta_1}{\sqrt{2\pi}}\mathrm{e}^{(1/2)\sigma^2\eta_1^2 t}\bigg\{\frac{\mathrm{e}^{\eta_1(h\sigma+\mu t)}}{\eta_1}\mathrm{Hh}_0\Big(\frac{h\sigma+\mu t}{\sigma\sqrt{t}}+\eta_1\sigma\sqrt{t}\Big)\\
&\qquad+\Big(\frac{1}{\sigma\eta_1\sqrt{t}}\Big)\sigma\sqrt{t}\sqrt{2\pi}\exp\Big\{(-\eta_1\sigma\sqrt{t})\sigma\eta_1\sqrt{t}+\frac{1}{2}\eta_1^2\sigma^2 t\Big\}\\
&\qquad\times\Phi\Big(\frac{h\sigma+\mu t}{\sigma\sqrt{t}}+\eta_1\sigma\sqrt{t}-\eta_1\sigma\sqrt{t}\Big)\bigg\}\\
&\quad= \mathrm{e}^{-\lambda t}\bigg\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{(1/2)\sigma^2\eta_1^2 t}\mathrm{e}^{\eta_1(h\sigma+\mu t)}\mathrm{Hh}_0\Big(\frac{h}{\sqrt{t}}+\Big(\frac{\mu}{\sigma}+\eta_1\sigma\Big)\sqrt{t}\Big)\\
&\qquad+\Phi\Big(\frac{h}{\sqrt{t}}+\frac{\mu}{\sigma}\sqrt{t}\Big)\bigg\}.
\end{aligned}
$$

Thus, its Laplace transform is given by

$$
\begin{aligned}
\int_0^\infty \mathrm{e}^{-\alpha t}\mathrm{e}^{-\lambda t}\bigg\{&\frac{1}{\sqrt{2\pi}}\mathrm{e}^{(1/2)\sigma^2\eta_1^2 t}\mathrm{e}^{\eta_1(h\sigma+\mu t)}\mathrm{Hh}_0\Big(\frac{h}{\sqrt{t}}+\Big(\frac{\mu}{\sigma}+\eta_1\sigma\Big)\sqrt{t}\Big)\\
&+\frac{1}{\sqrt{2\pi}}\mathrm{Hh}_0\Big(-\frac{h}{\sqrt{t}}-\frac{\mu}{\sigma}\sqrt{t}\Big)\bigg\}\,\mathrm{d}t.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{(1/2)\sigma^2\eta_1^2 t}\mathrm{e}^{\eta_1(h\sigma+\mu t)}\mathrm{Hh}_0\Big(\frac{h}{\sqrt{t}}+\Big(\frac{\mu}{\sigma}+\eta_1\sigma\Big)\sqrt{t}\Big)\,\mathrm{d}t\\
&\quad= \mathrm{e}^{\eta_1 h\sigma}\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\{(-\alpha-\lambda+\tfrac{1}{2}\sigma^2\eta_1^2+\eta_1\mu)t\}\,\mathrm{Hh}_0\Big(\frac{h}{\sqrt{t}}+\Big(\frac{\mu}{\sigma}+\eta_1\sigma\Big)\sqrt{t}\Big)\,\mathrm{d}t\\
&\quad= \mathrm{e}^{\eta_1 h\sigma} H_0(h, \Upsilon_\alpha, c_+; 0)
\end{aligned}
$$

and

$$
\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t}\frac{1}{\sqrt{2\pi}}\mathrm{Hh}_0\Big(-\frac{h}{\sqrt{t}}-\frac{\mu}{\sigma}\sqrt{t}\Big)\,\mathrm{d}t = H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; 0\Big),
$$

the contribution of the first term in (B.4) to the Laplace transform is given by

$$
\mathrm{e}^{\eta_1 h\sigma} H_0(h, \Upsilon_\alpha, c_+; 0) + H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; 0\Big).
$$

The second term in (B.4) will contribute

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\frac{\lambda^n}{n!}\bar{P}_{n,j}\\
&\quad\times\int_0^\infty \mathrm{e}^{-(\alpha+\lambda)t}t^n\frac{(\sigma\sqrt{t}\eta_1)^j}{\sigma\sqrt{2\pi t}}\mathrm{e}^{(1/2)(\eta_1\sigma)^2 t}I_{j-1}\Big(-h\sigma-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\Big)\,\mathrm{d}t
\end{aligned}
$$

(note that $a - b = -h\sigma$). The third term in (B.4) will contribute

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{Q}_{n,j} \times \int_0^{\infty} \mathrm{e}^{-(\alpha+\lambda)t} t^n \frac{(\sigma\sqrt{t}\eta_2)^j}{\sigma\sqrt{2\pi t}} \mathrm{e}^{(1/2)(\eta_2\sigma)^2 t} I_{j-1}\Big(-h\sigma - \mu t; \eta_2, \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\Big)\,\mathrm{d}t.$$

Therefore, we have, by Lemma B.4,

$$\begin{aligned}
\int_0^{\infty} &\mathrm{e}^{-\alpha t}\,\mathrm{P}(X_t + \xi^+ \geq a - b)\,\mathrm{d}t \\
&= \mathrm{e}^{\eta_1 h\sigma} H_0(h, \Upsilon_\alpha, c_+; 0) + H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; 0\Big) \\
&\quad + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\Big(\sum_{j=1}^{n+1}\bar{P}_{n,j} + \sum_{j=1}^{n}\bar{Q}_{n,j}\Big) H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) \\
&\quad + \mathrm{e}^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\frac{\lambda^n}{n!}\bar{P}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n) \\
&\quad - \mathrm{e}^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{Q}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h, \Upsilon_\alpha, c_-; n) \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!} H_0\Big(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n\Big) + \mathrm{e}^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{P}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n) \\
&\quad - \mathrm{e}^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{Q}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h, \Upsilon_\alpha, c_-; n) \\
&\quad + \mathrm{e}^{\eta_1 h\sigma} H_0(h, \Upsilon_\alpha, c_+; 0) + \mathrm{e}^{h\sigma\eta_1}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!} p^n \sum_{i=0}^{n}(\sigma\eta_1)^i H_i(h, \Upsilon_\alpha, c_+; n),
\end{aligned}$$

where the last equality follows from the facts that $\sum_{j=1}^{n+1}\bar{P}_{n,j} + \sum_{j=1}^{n}\bar{Q}_{n,j} = 1$ and $\bar{P}_{n,n+1} = p^n$.

## Acknowledgement

This work was supported by NSF grants DMS0074637, DMI9908106, DMS0103669, and DMS0216979.

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