Maximum Likelihood Estimation of Asymmetric Jump-Diffusion Processes: Application to Security Prices*

CYRUS A. RAMEZANI

Finance Department

Orfalea College of Business

California Polytechnic

San Luis Obispo, CA 93407

YONG ZENG

Department of Mathematics and Statistics

University of Missouri, Kansas City, MO 64110

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ABSTRACT

An asymmetric jump-diffusion model of stock price behavior is proposed. In an extension of Merton (1976*b*), we posit that returns dynamics are determined by a drift component, a Wiener process and two jump processes representing the arrival of "good" or "bad" news that lead to jumps in security prices. We assume that good and bad news may arrive with different intensities and the distribution of jump magnitudes representing each type is different. To admit and test these distinctions, we assume that news arrives according to two Poisson processes and jump magnitudes representing good and bad news are Pareto and Beta distributed. We develop cumulant and maximum likelihood estimators and use daily stock prices data to estimate the proposed model. Empirical results strongly support the posited model. Likelihood based test provides support to the hypothesis that stock prices respond differently to the arrival of good and bad news.

Keywords: Asset Price Processes, Jump-Diffusion Models, MLE, Leptokurtic Distributions

JEL Classification: C13, C22, G12, G13

Introduction

Portfolio choice and asset valuations models of modern finance theory critically depend upon the form of the probability distribution describing security-price changes. The log-normal distribution with constant parameters is the most convenient and widely adopted form. A large body of evidence, however, shows that the log-normal model fits actual returns data rather poorly, primarily because empirical return distributions exhibit excess kurtosis and skewness relative to the normal distribution.[1]

The failure of the log-normal model has led to alternative characterization of the price processes. The proposed alternatives can be categorized into three groups. The first class of models posit linear price processes that lead to finite- and infinite-variance distributions. The stable Paretian distribution proposed by Mandelbrot (1963) and Fama (1965) is the main example of an infinite variance distribution. The finite-variance models arise from the student-t distribution or mixture of distributions such as Poisson-normal (jump-diffusion) and the compound normal.[2] Second, the Auto Regressive Conditional Heteroscedastic (ARCH) model, and its numerous extensions surveyed by Bollerslev, Chou & Kroner (1992) refute the assumption that the return process is stationary. These models posit that the conditional first and second moments of stock returns are time varying and persistent, particularly over long horizons. The third class of models combines jump and ARCH effects, where volatility is driven by small random shocks through an ARCH process and the occurrence of a jump event can either break the persistence in the volatility process or directly impact the return process.[3]

The model we posit belongs to the class of mixed jump-diffusion processes, first proposed by Roberts (1959) and Press (1967) and later extended by Merton (1976*b*). Using daily return data on stocks and indices, Beckers (1981), Ball & Torous (1983), Jarrow & Rosenfeld (1984), Jorion (1988), Tucker (1992), and Das & Sundaram (1999) provide empirical evidence in support of the jump-diffusion model.[4] The existence of a jump component in security prices is also supported by a large body of evidence from "event studies." In particular, the evidence presented in Ederington & Lee (1993) documents the fact that information flows associated with firm-specific and macroeconomics factors lead to large movements in asset prices.

In the standard jump-diffusion model (Merton 1976*b*), the returns process consists of three components, a linear drift, a Brownian motion representing "normal" price vibrations, and a compound Poisson process that accounts for "abnormal" change in prices (jumps) due to arrival of "news". The discrete points in time when news arrives are assumed to be random and driven by a Poisson process. Upon the arrival of "news," jump magnitudes are determined by sampling from an independent and identically distributed (IID) random variable. Merton further assumes that the logarithm of jump magnitude is normally distributed. This special case makes estimation and hypothesis testing tractable and has become the most important representation of the jump-diffusion model. Ball & Torous (1983), Jarrow & Rosenfeld (1984), Bates (1991), Tucker (1992), and others employ this special case.[5]

Since there is only one jump component in the standard jump-diffusion model, news that cause upward jump in prices –"good news"– and news that cause downward jump in prices –"bad news"– are not distinguished by their intensity or distributional characteristics.[6] This potential limitation of the simple jump-diffusion model provides the main motivation for our model. We conjecture that the arrival frequency and the distributional characteristics of jumps representing "good" and "bad" news are different. We propose a jump-diffusion model that permits this type of information arrival and use maximum likelihood procedures to estimate its parameters.

There are several economic justifications for the proposed model. At a microeconomic level, Milgrom (1981) has formalized the notion of "good" and "bad" news and shown that such distinction plays an important role in rational expectation models that are the foundation of information economics. In particular, Milgrom (1981) shows that the arrival of good (bad) news about a firm's prospects always leads to a rise (fall) in its share price. Our characterization of "good" and "bad" news is consistent with his representation theorems. At the macroeconomics level, expansionary and contractionary periods are accompanied with unequal frequency of good and bad news arrivals. The differential in intensity of news arrival may be in turn driven by broader economic cycles, driven by recurrent technological change and innovation, or perhaps unexpected shifts in social, demographic, and political cycles, such as the congressional and presidential elections.[7]

In the reminder, we first present our model in detail. Maximum likelihood estimation of the proposed model is then discussed. The model is estimated using firm specific data from the U.S. Using the Geometric Brownian Motion (GBM) as the null, we test the single and double jump models as alternatives and provide evidence favoring jump models. We conclude by considering the implications of our findings and discussing directions for further research.

The Model

The model we propose is a generalizations of Merton's single jump-diffusion model. The motivation is to introduce a distinction between "good" and "bad" news, defined as any surprise information that leads to non-marginal price increases (up-jumps) or decreases (down-jumps) respectively. That is, we have two independent Poisson processes generating good and bad news. This separation of good from bad news implies that the range of values for the random percentage change in price must be constrained. Because stocks represent limited liability, percentage change in prices due to bad news must be bounded from below by minus one hundred percent. Similarly, the percentage change in prices due to arrival of good news must be positive.

Because of these constraints we cannot assume a log-normal distribution for either up or down jump magnitudes. Instead we assume that jump magnitudes representing good news are distributed according to a Pareto distribution and jump magnitudes representing bad news are Beta distributed. Though the boundaries on percentage price change limits our distributional choice, we show that the selected distributions lead to tractable maximum likelihood estimation.

Let $S(t)$ denote the price of stock at time t and assume that the price process can be represented by the following model:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t) + \sum_{j=u,d} (Y^j_{N^j(\lambda^j t)} - 1) dN^j(\lambda^j t) \tag{1a}$$

where μ and σ are the drift and volatility terms, Z(t) is a standard Wiener process, $Y^j, j = u, d$, is the jump magnitude, and $N^j(\lambda^j)$ are independent Poisson processes with intensity parameters λ^j (u and d represent up- and down-jumps respectively).

We assume that the up-jump magnitudes (Y^u) are distributed Pareto(r_u) with density function $f_{Y^u}(Y^u)$=$r_u(\frac{1}{Y^u})^{r_u+1}$ where $Y^u \geq 1$, E(Y^u)=$\frac{r_u}{r_u-1}$ and var(Y^u)=$\frac{r_u}{(r_u-2)(r_u-1)^2}$. Similarly, the down-jump magnitudes (Y^d) are distributed Beta(r_d,1) with density function $f_{Y^d}(Y^d) = r_d(Y^d)^{r_d-1}$ where $0 < Y^d < 1$, E(Y^d) $= \frac{r_d}{(r_d+1)}$ and var(Y^d) $= \frac{r_d}{(r_d+2)(r_d+1)^2}$. All jump magnitudes Y_js are assumed to be independent. Henceforth we will refer to equation (1a) as the Pareto-Beta Jump-Diffusion (PBJD) model.

Equation (1a) is a stochastic differential equation, offering a particular description of stock price dynamics: The total change in stock price is due to a deterministic growth rate per unit of time (μ) plus three independent stochastic components. The first source of randomness is "local" price variations that are due to normal clearing of imbalances in demand and supply. The second (third) source of randomness is due to the random arrival of good (bad) news which is driven by a Poisson process and leads to "abnormal" upward (downward) movement in price.

The Doléans-Dade formula (Protter 1991) provides an explicit solution for (1a):

$$S(t) = S(0)\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z(t)\} \prod_{j=u,d} Y^j(N(\lambda^j t)) \tag{1b}$$

where for $j = u, d$,

$$Y^j(N(\lambda^j t)) = \begin{cases} 1 & \text{if } N(\lambda^j t) = 0 \\ \prod_{i=1}^{N(\lambda^j t)} Y_i^j & \text{if } N(\lambda^j t) = 1, 2, 3, \ldots \end{cases}$$

and $N(\lambda^j t)$, are Poisson distributed with parameter $\lambda^j t$. Using equation(1b), the s period rate of return, $r(s)$, is:

$$r(s) = (\mu - \frac{1}{2}\sigma^2)s + \sigma Z(s) + \sum_{i=1}^{N_s^u} ln(Y_i^u) + \sum_{i=1}^{N_s^d} ln(Y_i^d) \tag{1c}$$

where the number of good (bad) news over the time period s, N_s^js, are independent Poisson distributed random variables with parameters λ^js.

Merton's Jump-Diffusion (JD) model has a single jump component with magnitude Y distributed IID log-normal (α, β^2) and Poisson (λ) arrival rate. The PBJD collapses to a single JD model when $\lambda = \lambda_u + \lambda_d$ and the jump magnitude has a mixed distribution of Pareto(r_u) with probability $\frac{\lambda_u}{\lambda_u+\lambda_d}$ and Beta(r_d,1) with probability $\frac{\lambda_d}{\lambda_u+\lambda_d}$.[8] It is important to note, however, that JD and PBJD *are not* nested. Without the jump components, the two models reduce to the standard Geometric Brownian Motion (GBM).

The Conditional Density

Let $N_s^u = m$ and $N_s^d = n$ be the number of up- and down-jumps during the time interval $t = 0$ to s. The conditional densities of s period returns can be derived under four combination of m and n: $m = 0$ and $n = 0$ (no jumps occur); $m = 0$ and $n \geq 1$ (only down-jumps occur); $m \geq 1$ and $n = 0$ (only up-jumps occur); and $m \geq 1$ and $n \geq 1$ (both types of jumps occur). All four conditional densities can be derived using convolution techniques and distributional properties. Before deriving the conditional density, we state some useful facts about Pareto, Beta and exponential distributions (Patel, Kapadia & Owen 1976):

F1. If $Y^u \sim$ Pareto (r_u), then $\ln(Y^u) \sim \exp(r_u) = \Gamma(1, r_u)$.
F2. If $Y^d \sim$ Beta $(r_d, 1)$, then $-\ln(Y_d) \sim \exp(r_d) = \Gamma(1, r_d)$.
F3. If $X = Y_1 + Y_2 + \cdots + Y_n$, where $Y_i \sim \exp(\theta)$ and they are independent, then $X \sim \Gamma(n, \theta)$.

Let $U = \sum_{i=1}^{N_s^u} ln(Y_i^u) > 0$, $D = \sum_{i=1}^{N_s^d} ln(Y_i^d) < 0$ and $T = U + D$. Then s period return can be written as $r(s) = (\mu - 0.5\sigma^2)s + Z(s) + U + D$. For $N_s^u = m \geq 1$ the conditional distribution of U (by F1 and F3) is $U|m \sim \Gamma(m, r_u)$:

$$f_{U|m}(U) = \frac{r_u^m}{(m-1)!} U^{m-1} e^{-r_u U}$$

Similarly, for $N_s^d = n \geq 1$ the conditional distribution of D is $-D|n \sim \Gamma(n, r_d)$:

$$f_{D|n}(D) = \frac{r_d^n}{(n-1)!} (-D)^{n-1} e^{r_d D}$$

Using these results the conditional density of $T = U + D$, given $m \geq 1$ and $n \geq 1$, is:

$$\begin{aligned} f_{T|m,n}(t) &= \int_{-\infty}^{\infty} f_D(x) f_U(t-x) dx \\ &= \frac{r_u^m r_d^n e^{-r_u t}}{(m-1)!(n-1)!} \int_{-\infty}^{0 \wedge t} (-x)^{n-1} (t-x)^{m-1} e^{(r_u + r_d)x} dx \end{aligned} \tag{2}$$

Now, we are ready to determine all four conditional densities. For the case $m = 0$ and $n = 0$, the conditional density is that of GBM:

$$f_{r(s)|0,0}(r) = \frac{1}{\sqrt{2\pi s}\sigma} e^{-\frac{1}{2\sigma^2 s}(r - \mu s + 0.5\sigma^2 s)^2} \tag{3a}$$

When $m = 0$ and $n \geq 1$, the conditional distribution is the independent sum of $-\Gamma(n, r_d)$ and $\mathrm{N}((\mu - \frac{1}{2}\sigma^2)s,\, \sigma^2 s)$:

$$f_{r(s)|0,n}(r) = \frac{r_d^n}{(n-1)!\sqrt{2\pi s}\sigma} \int_{-\infty}^{0} (-x)^{n-1} e^{r_d x - \frac{1}{2\sigma^2 s}(r-x-\mu s+0.5\sigma^2)^2 s} dx \tag{3b}$$

Similarly, for $m \geq 1$ and $n = 0$, the conditional distribution is the independent sum of $\Gamma(m, r_u)$ and $\mathrm{N}((\mu - \frac{1}{2}\sigma^2)s,\, \sigma^2 s)$:

$$f_{r(s)|m,0}(r) = \frac{r_u^m}{(m-1)!\sqrt{2\pi s}\sigma} \int_{0}^{\infty} (x)^{m-1} e^{-r_u x - \frac{1}{2\sigma^2 s}(r-x-\mu s+0.5\sigma^2 s)^2} dx \tag{3c}$$

Finally, for $m \geq 1$ and $n \geq 1$, the conditional distribution is the independent sum of the distribution for T and $\mathrm{N}((\mu - \frac{1}{2}\sigma^2)s,\, \sigma^2 s)$. Then the conditional density of $r(s)$ is:

$$\begin{aligned} f_{r(s)|m,n}(r) = {} & \frac{r_u^m r_d^n}{(m-1)!(n-1)!\sqrt{2\pi s}\sigma} \\ & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{0 \wedge t} (-x)^{n-1}(t-x)^{m-1} e^{(r_u+r_d)x} dx \right) \times e^{-r_u t} e^{-\frac{1}{2\sigma^2 s}(r-t-\mu s+0.5\sigma^2 s)^2} dt \end{aligned} \tag{3d}$$

The Unconditional density

Next we derive the unconditional density of $s = 1$ period returns. This function plays a critical role for estimation and hypothesis testing. Letting $P(n, \lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$, the unconditional density for one period returns, $f(r)$, can be written as the Poisson weighted sum of the four conditional densities (3a-3d):

$$f(r) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P(n, \lambda_d) P(m, \lambda_u) f_{n,m}(r)$$

$$\begin{aligned} &= e^{-(\lambda_u+\lambda_d)} f_{0,0}(r) + e^{-\lambda_u} \sum_{n=1}^{\infty} P(n, \lambda_d) f_{0,n}(r) \\ &\quad + e^{-\lambda_d} \sum_{m=1}^{\infty} P(m, \lambda_u) f_{m,0}(r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P(n, \lambda_d) P(m, \lambda_u) f_{n,m}(r) \end{aligned} \tag{4}$$

The second expression in (4) shows that the unconditional distribution of returns is a *mixture density*. This fact has important implications for the maximum likelihood estimation of the process, which will be discussed later.

Table (1) presents the expression for the first four moments of $r(s)$ under GBM, JD, and PBJD models.[9] Clearly, relative to GBM, both the JD and the PBJD have positive kurtosis leading to a leptokurtic returns' distribution. Whether JD or PBJD provides a better description of data is addressed in the empirical section presented next.

Estimation and Hypothesis Testing

The proposed PBJD model is a first-order stochastic differential equations of generalized Itô type. Such processes can be estimated by maximum likelihood estimation (MLE), the method of moments and its variants including cumulant matching, generalized method of moments (GMM), and simulated moment estimation.[10] The cumulant and MLE methods will be used in this study.

For large samples, MLE is the best method of estimation, because under mild regularity conditions, the estimated parameter are consistent, asymptotically normal and asymptotically efficient (Basawa & Rao 1980, Brown & Hewitt 1978, Lo 1988, Sorensen 1991). However, MLE requires a complete specification of the transition density, which for nonlinear models may not have an explicit expression. Fortunately the PBJD is a linear process with independent increments and explicit transition density. Moreover, the selected distributions for the jump components have properties that make the MLE tractable.

Let $S(0), S(1), S(2), \ldots, S(M)$ denote realizations of stock price at equally-spaced times $k = 0, 1, 2, \ldots, M$. The one period rate of return $r(t) = \ln S(t) - \ln S(t-1)$ is IID with the density function (4). Hence the estimation problem at hand has the classical IID set-up.

One method for obtaining parameter estimates for our model is to "match cumulants." We choose this method rather than "moment matching" because the cumulants of $r(s)$ are easier to compute . PBJD has six unknown parameters $(\mu, \sigma^2, \lambda_u,\ r_u, \lambda_d, r_d)$ which would require the matching of the first six population and sample cumulants. Solving the resulting equations for the parameters provides a set of estimates. This method was employed by Press (1967), Beckers (1981), and Ball & Torous (1983) to obtain parameter estimate for the JD model.

Appendix (1) contains the derivation of the cumulants for PBJD model. Estimation by cumulant matching yields consistent, but inefficient estimators. Moreover, because cumulants are functions of the sample moments, the distributions of the cumulant estimators in large samples will be normal (Press 1968). However, the cumulant estimates may not exist, or have the wrong sign. This drawback has limited the usefulness of cumulant matching in empirical work. Given these limitation, we use cumulant matching as a means to obtain initial values for MLE.

The developments in maximum likelihood estimation of Itô processes is discussed in Lo (1988) and Sorensen (1991). They prove the consistency, asymptotic normality and asymptotic efficiency of MLE.[11] With equally-spaced sampled data, the log-likelihood given M returns observations is:

$$L(r; \lambda_u, \lambda_d, r_u, r_d, \mu, \sigma^2) = \sum_{i=1}^{M} ln(f(r_i)) \tag{5}$$

where $f(r_i)$ is given by (4). As noted, the unconditional distribution of returns is a mixture density. Kiefer (1978) has shown that for mixture densities like (4), a global maximum of the log-likelihood function (5) does not exist. This is because a singularity arises when for the ith observation $r_i = \mu$ and $\sigma_i^2 \to 0$. At such point the log-likelihood function (5) becomes infinite.[12]

Kiefer (1978) proved that if the parameter space is compact and large enough to include the true parameters, then a bounded local maximum of the likelihood function exists and the parameter estimates will be consistent and asymptotically normal. Moreover, standard errors for the estimates can be constructed by standard procedures such as using the information matrix. Hamilton (1994, page 689) and Kiefer (1978) have offered remedies to deal with the singularity problem associated with the mixture density. As Hamilton (1994) shows singularities do not pose a major problem so long as the the selected numerical maximization procedure converges to a local maxima.

The Newton-Raphson method has been the most widely used numerical procedure for jump-diffusion models. This optimization method requires the first and second order derivatives of the log-likelihood function. Such derivatives are difficult to compute for the PBJD model. To avoid this difficulty we use Powell's method. The latter method uses successive line optimization in the conjugate directions and does not necessitate the use of the derivatives.[13]

It is straight forward to show that the regularity conditions described in Kiefer (1978) generalize to finitely many mixed distributions such as (4). This is particularly true since we truncate the Poisson compound sums (see below). To avoid the singularity problem we choose a range of initial values to ensure the parameter space is large enough to cover the true parameter values. We also ensure that the likelihood function obtained by Powell's method converges. Hence the conditions described in Hamilton (1994) and Kiefer (1978) are met and the consistency and asymptotic normality of the obtained maximum likelihood estimates are guaranteed.

The likelihood function in (4) involves double infinite summations and double improper integrals. First, piecewise Gaussian quadratures are employed to compute the integral and the double integral (Press, Teukolsky, Vetterling & Flannery 1992). We find that for plausible parameter values, lower bound truncation of the integrals at (-2.0) provides six digit accuracy. Next, the infinite sums are calculated using the usual termination criterion; if $S_n = \sum_{i=1}^{n} X_i$, then we stop the summation if $2|X_{n+1}| \leq FTOL \times (|S_n| + |S_{n+1}|)$ (Press et al. 1992). We choose $FTOL = 10^{-10}$, which will guarantees at least eight digits accuracy. Standard error for the estimates is obtained by the *outer-product* method, which is based on the first derivative of the likelihood function (see Hamilton (1994), page 143).

Within the MLE framework, the likelihood ratio test (LRT) is the standard method for testing alternative hypothesis and under mild regularity conditions, the LRT has an asymptotic χ^2 distribution (see Basawa & Rao (1980) and Sorensen (1991)). All empirical papers in this area employ LRT to test the validity of JD versus the GBM model and the results generally favors the JD model. Our null hypothesis is also GBM with alternatives being JD or PBJD. Under the null and the alternative(s), the log-likelihoods for M observations of daily returns are, respectively:

$$L_0(r;\theta_0) = \sum_{i=1}^{M} ln(f_{0,0}(r_i))$$

$$L_1(r;\theta_1) = \sum_{i=1}^{M} ln(f(r_i))$$

where $\theta_0 = \{\hat{\mu}_0, \hat{\sigma}_0^2\}$ and $\theta_1 = \{\hat{\mu}_1, \hat{\sigma}_1^2, \hat{\lambda}_u, \hat{r}_u, \hat{\lambda}_d, \hat{r}_d\}$ are parameter estimates obtained from maximum likelihood estimation of each model. Under the assumed regularity conditions, $-2\ln(\Lambda) = -2\,[L_0(r;\theta_0) - L_1(r;\theta_1)]$ is asymptotically Chi-squared distributed with 4 degrees of freedom.[14]

Data and Results

The Data are 507 daily returns for six New York Stock Exchange (NYSE) listed stocks (log-relatives adjusted for dividend and stock splits spanning the period 1/1/1991 to 12/31/1992).[15] The firms are Boeing (BA), Bethlehem Steel (BS), Delta Airlines (DAL), Ford Motors (F), Goodyear Tires (GT), and International Business Machines (IBM).

Table (2) presents the sample statistics for the data used and figures (1) through (6) contain time-series plots. The six series exhibit varying degree of volatility, positive and negative trend, and cyclical patterns during the period considered. For example, visual inspection of Figures (1), (2), and (5) indicates a downward trend common to these series but different frequencies of large price changes (greater than 5% in absolute value). In general all six series exhibit large negative and positive prices changes that occur with different intensity, all of which is consistent with the skewness and excess kurtosis indicated by the sample statistics in Table (2).

Table (3) presents the MLE estimates for the parameters of the GBM model along with the value for the log-likelihood function. Drift and volatility estimate vary widely across the series. As expected, MLE estimates of μ and σ are highly significant (standard error appears below the parameter estimate). The annualized expected return estimates range from -34.95% (IBM) to well over 100%. Similarly annualized volatility estimates range from 25% to 43%.[16]

The parameter estimates for the JD model are reported in Table (4). Relative to GBM the log-likelihood values increase significantly. The LRT statistics indicates that for all series the null hypothesis of no-jump component can be rejected at 1% significance level, which is consistent with findings reported in the extant empirical literature (Das & Sundaram 1999, Jorion 1988, Tucker 1992). However, as noted earlier, these results should be interpreted carefully since the test is performed on the boundary of the parameter space ($\lambda_j = 0, j = u, d$).

One alternative to the JD model commonly tested in the literature restricts the mean jump magnitude (α) to equal zero, which in turn leads to a symmetric returns distribution (zero skewness). Table (5) report MLE parameters for that model and two sets of LRT results. The first LRT tests the null hypothesis of GBM against the alternative of JD model with zero mean jump. This hypothesis is rejected at 1% significance level for all series, indicating that the jump component plays an important role in determining returns dynamics. The second LRT tests the hypothesis of zero versus non-zero mean jump magnitude.[17] The last column of Table (5) shows that this hypothesis cannot be rejected for all stocks considered.

Considering both JD models, it is apparent that the addition of a jump component significantly changes the estimated drift and volatility parameters associated with the continuous part of the process. However, without restricting the mean jump magnitude to equal zero, some of the Poisson intensity parameters (BS, F, and GT in Table 4) are too large to be consistent with the notion that non-marginal jump in prices are "rare" events that occur infrequently. This type of finding has been reported in most empirical examination of the JD model.

The cumulant estimates for the PBJD model are presented in Table (6). Press (1968) and Beckers (1981) reported non-existence or negative volatility estimates for the JD model. For our sample, the cumulant method produces reasonable result for only three stocks. Furthermore, our volatility estimates are consistently positive. The cumulant estimates, when available, are used as initial values for the maximum likelihood estimation.

Table (7) contain the MLE parameter estimates for our model. Unlike the cumulant method, MLE produces reasonable estimates for all series. The caveat regarding tests of hypothesis on the boundary of parameter space withstanding, the null hypotheses of GBM is rejected at 0.01 significance level for all stocks. The significant increase in the log-likelihood values provide strong support for PBJD model.

The parameter estimate in Table (7) in conjunction with the formulas in Table (1) can be used to calculate the moments of the returns distribution. For example, using the estimated parameters one can obtain the value of the first four moments while distinguishing the effect of the jumps from the continuous components of the process.

The PBJD model offers clear improvements in characterizing the empirical distribution of returns. This parameterization of the return process can serve as the starting point for furthering research in a number of areas in economics and finance. In particular, asset and option pricing models proposed by Bates (1991), Jarrow & Rosenfeld (1984), Naik & Lee (1990), and Merton (1976*b*) can be extended to the PBJD process. Intertemporal portfolio (consumption) choice models can be enhanced in similar vein. For a variety of other economic variables, including foreign exchange, inflation, and short term interest rates, information arrival plays a significant role in driving the dynamics of the process. Our model can be easily adopted to these areas as well. Moreover it may be interesting to assess the significance of the proposed distinction between good and bad news in these settings.

Conclusions

The paper presented an asymmetric jump-diffusion model for security prices. The cumulant and maximum likelihood estimation procedures were used to estimate the parameters of this model. The model was applied to daily data for six NYSE listed stocks. Likelihood ratio tests for alternative hypothesis were implemented. Empirical evidence strongly supports the proposed model.

There are a number of interesting directions for future extensions of this work. As a starting point, optimal portfolio choice rules can be derived taking the proposed returns process as exogenous. Such exercise will make it possible to derive asset pricing functions, including an option pricing formula. Following Merton (1976*a*), the problem of errors in option pricing due to the misspecification of the stochastic process generating the underlying stock returns can also be addressed using our model. In related research we are currently investigating some of these issues.

Appendix 1: Cumulants Computation for PBJD Model

The derivation of the moments and cumulants of PBJD model presented below follows statistical procedures presented in Kendall & Stuart (1977). Let $\phi_X(s) = E(e^{Xs})$ be the moment generating function (MGF) of random variable X. Then the cumulant generating function (CGF) is defined as $\kappa_X(s) = \ln \phi_X(s)$, and the cumulants, $\kappa_1, \kappa_2, \kappa_3, \dots, k_r$ are defined by $\kappa_X(s) = \ln \phi_X(s) = \sum_{i=1}^{\infty} \frac{\kappa_i}{i!} t^i$. Cumulants and moments are related by the following relationship:

$$\kappa_1 t + \frac{\kappa_2 t^2}{2!} + \cdots + \frac{\kappa_i t^i}{i!} + \cdots = \ln(1 + (EX)t + \frac{(EX^2)t^2}{2!} + \cdots + \frac{(EX^i)t^i}{i!} + \cdots)$$

We use this relationship and the following results to derive cumulant estimators for PBJD model.

1. Cumulants are additive: Suppose κ_i^X is the ith cumulant of X, and κ_i^Y is the ith cumulant of Y, and X and Y are independent. Let $Z = X + Y$. Then the ith cumulant of Z is $\kappa_i^Z = \kappa_i^X + \kappa_i^Y$.

2. Cumulants of Compound Poisson Process: Suppose $Y = \sum_{i=1}^{N(\lambda t)} X_i$, where X_i's are IID with MGF $\phi_X(s)$ and $N(\lambda t)$ is Poisson process with intensity λ. Then MGF of Y is $\phi_Y(s) = \exp\{\lambda t(\phi_X(s) - 1)\}$. The CGF for this process is $\kappa_Y(s) = \lambda t(\phi_X(s) - 1) = \lambda t \sum_{i=1}^{\infty} \frac{EX^i}{i!} s^k$. The r-th cumulant of Y is $\kappa_r = \lambda t E Y^r$ for $r = 1, 2, 3, \dots$.

3. Moments of the Exponential Distribution: Let X has an exponential density, $f_X(x) = \theta e^{-\theta x}$. Then the r-th moment $EX^r = \frac{r!}{\theta^r}$.

The first six cumulants of PBJD can be computed from sample moments as follows:

$$
\begin{aligned}
\bar{K}_1 &= m_1, \\
\bar{K}_2 &= m_2 - m_1^2, \\
\bar{K}_3 &= m_3 - 3m_2m_1 + 2m_1^3, \\
\bar{K}_4 &= m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4 \\
\bar{K}_5 &= m_5 - 5m_4m_1 - 10m_3m_2 + 20m_3m_1^2 + 30m_2m1^3 - 60m_2m_1^3 + 24m_1^5 \\
\bar{K}_6 &= m_6 - 6m_5m_1 - 15m_4m_2 + 30m_4m_1^2 - 10m_3^2 + 120m_3m_2m_1 - 120m_3m_1^3 \\
&\quad + 30m_2^3 - 270m_2^2m_1^2 + 360m_2m_1^4 - 120m_1^6
\end{aligned}
$$

where $\bar{K}_h, h = 1, \cdots, 6$, denote the sample cumulants and $\bar{m}_h, h = 1, \cdots, 6$, denote the sample moments. The first six population cumulants are:

$$
\begin{aligned}
K_1 &= s(\mu - \frac{1}{2}\sigma^2 + \frac{\lambda_u}{r_u} - \frac{\lambda_d}{r_d}) & K_2 &= s(\sigma^2 + 2\frac{\lambda_u}{r_u^2} + 2\frac{\lambda_d}{r_d^2}) \\
K_3 &= 6s(\frac{\lambda_u}{r_u^3} - \frac{\lambda_d}{r_d^3}) & K_4 &= 24s(\frac{\lambda_u}{r_u^4} + \frac{\lambda_d}{r_d^4}) \\
K_5 &= 120s(\frac{\lambda_u}{r_u^5} - \frac{\lambda_d}{r_d^5}) & K_6 &= 720s(\frac{\lambda_u}{r_u^6} + \frac{\lambda_d}{r_d^6})
\end{aligned}
$$

Setting $K_h = \bar{K}_h, h = 1, \ldots, 6$, yields the cumulant estimates,

$$
\begin{aligned}
0 &= (\frac{\bar{K}_5^2}{5} - \frac{\bar{K}_4\bar{K}_6}{6})\frac{\hat{r_u}^2}{20} + (\frac{\bar{K}_4\bar{K}_5}{2} + \frac{\bar{K}_3\bar{K}_6}{3})\frac{\hat{r_u}}{10} + (\frac{\bar{K}_4^2}{4} - \frac{\bar{K}_3\bar{K}_5}{5}) \\
\hat{r_d} &= \frac{5\bar{K}_4\hat{r_u} - 20\bar{K}_3}{-\bar{K}_5\hat{r_d} + 5\bar{K}_4} \\
\hat{\lambda_d} &= \frac{\hat{r_d}^4(\frac{\bar{K}_4}{24s}r_u - \frac{\bar{K}_3}{6s})}{\hat{r_u} + \hat{r_d}} \\
\hat{\lambda_u} &= \hat{r_u^3}(\frac{\bar{K}_3}{6s} + \frac{\hat{r_d}(\frac{\bar{K}_4}{24s}\hat{r_u} - \frac{\bar{K}_3}{6s})}{\hat{r_u} + \hat{r_d}}) \\
\hat{\sigma^2} &= \frac{\bar{K}_2}{s} - 2\frac{\hat{\lambda_u}}{\hat{r_u^2}} - 2\frac{\hat{\lambda_d}}{\hat{r_d^2}} \\
\hat{\mu} &= \frac{K_1}{s} + \frac{1}{2}\hat{\sigma^2} - \frac{\hat{\lambda_u}}{\hat{r_u}} + \frac{\hat{\lambda_d}}{\hat{r_d}}
\end{aligned}
$$

That is, the cumulant estimators are obtained by first solving a quadratic equation for its positive root for r_u and then substituting to find the remaining five estimates.

Table 1: The First Four Moments for Alternative Models

	GBM	JD	PBJD
E$r(s)$	$(\mu - \frac{1}{2}\sigma^2)s$	$(\mu - \frac{1}{2}\sigma^2 + \lambda\alpha)s$	$(\mu - \frac{1}{2}\sigma^2 + \frac{\lambda_u}{r_u} - \frac{\lambda_d}{r_d})s$
Var$[r(s)]$	$\sigma^2 s$	$(\sigma^2 + \lambda(\beta^2 + \alpha^2))s$	$(\sigma^2 + 2\frac{\lambda_u}{r_u^2} + 2\frac{\lambda_d}{r_d^2})s$
Skewness	0	$\frac{\lambda\alpha(\alpha^2+3\beta^2)}{(\sigma^2+\lambda(\beta^2+\alpha^2))^{\frac{3}{2}}\sqrt{s}}$	$\frac{6(\frac{\lambda_u}{r_u^3}-\frac{\lambda_d}{r_d^3})}{(\sigma^2+2\frac{\lambda_u}{r_u^2}+2\frac{\lambda_d}{r_d^2})^{1.5}\sqrt{s}}$
Kurtosis	0	$\frac{\lambda(\alpha^4+6\alpha^2\beta^2+3\beta^4)}{(\sigma^2+\lambda(\beta^2+\alpha^2))^2 s}$	$\frac{24(\frac{\lambda_u}{r_u^4}+\frac{\lambda_d}{r_d^4})}{(\sigma^2+2\frac{\lambda_u}{r_u^2}+2\frac{\lambda_d}{r_d^2})^2 s}$

Table 2: Sample Statistics for Dividend-Adjusted Daily Stock Returns (507 Obs.)

Stock	Mean	Std Dev	Min	Max	Skewness	Kurtosis
BA	-2.402E-5	0.0161	-0.077	0.063	-0.050	2.67
BS	5.846E-4	0.0274	-0.141	0.152	0.383	3.08
DAL	8.046E-3	0.0194	-0.065	0.078	0.298	0.90
F	1.363E-3	0.0207	-0.073	0.104	0.385	1.56
GT	2.807E-3	0.0216	-0.073	0.115	0.611	2.65
IBM	-1.256E-3	0.0162	-0.107	0.061	-1.182	7.37

Table 3: ML Estimates for GBM Model

Stock	μ	σ	ln(lkhd)
BA	-2.402E-5 (0.715E-5)	0.0161 (0.0005)	1215.31
BS	5.846E-4 (0.122E-4)	0.0274 (0.0008)	1104.84
DAL	8.046E-3 (8.615E-4)	0.0194 (0.0006)	1279.65
F	1.363E-3 (0.919E-5)	0.0207 (0.0006)	1247.35
GT	2.807E-3 (0.941E-5)	0.0212 (0.0006)	1235.33
IBM	-1.256E-3 (0.715E-5)	0.0162 (0.0005)	1372.30

Standard Error in parantheses.

Table 4: ML Estimates for JD Model (Dividend-Adjusted Daily Data)

Stock	λ	α	β	μ	σ	ln(lkhd)	$-2\ln(\Lambda)$
BA	0.1617	3.791E-3	0.0254	-5.590E-4	0.0124	1396.79	47.42**
	(0.0521)	(4.286E-3)	(0.0031)	(7.203E-4)	(5.385E-4)		
BS	2.139	2.640E-3	0.0172	-5.054E-3	8.629E-3	1123.51	37.34**
	(0.3549)	(1.125E-3)	(0.0008)	(2.259E-3)	(2.768E-3)		
DAL	0.2239	7.832E-3	0.0202	-1.521E-3	0.0167	1286.33	13.36**
	(0.0556)	(4.452E-3)	(0.0034)	(1.201E-3)	(8.511E-4)		
F	3.746	1.757E-3	0.0101	-5.382E-3	4.220E-3	1260.94	27.18**
	(1.7300)	(8.014E-4)	(0.0006)	(2.987E-3)	(5.988E-3)		
GT	1.570	3.625E-3	0.0148	-2.823E-3	8.6603E-3	1255.18	39.70**
	(0.2058)	(1.095E-3)	(0.0008)	(1.602E-3)	(1.387E-5)		
IBM	0.0479	-0.0189	0.0407	-2.621E-4	0.0128	1422.71	100.82**
	(0.0495)	(1.268E-2)	(0.0075)	(5.460E-4)	(3.483E-4)		

'**' ('*') indicates significances at 1% (%5) level. Standard Error in parantheses.

Table 5: ML Estimates for JD Model with Mean Jump (α) Equal to Zero

Stock	λ	β	μ	σ	ln(lkhd)	$-2\ln(\Lambda_1)$	$-2\ln(\Lambda_2)$
BA	0.1557	0.0259	-2.421E-4	0.0125	1396.35	46.54**	0.88
	(0.0517)	(0.0032)	(6.419E-4)	(0.0005)			
BS	0.1068	0.0460	-3.662E-3	0.0228	1122.13	34.58**	2.76
	(0.0493)	(0.0059)	(1.133E-3)	(0.0009)			
DAL	0.1624	0.0241	-2.280E-6	0.0169	1284.87	10.44**	2.92
	(0.0522)	(0.0040)	(8.446E-4)	(0.0007)			
F	0.0975	0.0322	1.105E-3	0.0181	1256.32	17.94**	9.24**
	(0.0489)	(0.0053)	(8.786E-4)	(0.0007)			
GT	0.3942	0.0233	2.173E-3	0.0152	1252.13	33.60**	6.1*
	(0.0655)	(0.0021)	(8.783E-4)	(0.0009)			
IBM	0.0425	0.0473	-4.972E-4	0.0132	1420.77	96.94**	3.88*
	(0.0462)	(0.0102)	(6.273E-4)	(0.0004)			

'**' ('*') indicates significances at 1% (%5) level. Standard Error in parantheses.

Table 6: Cumulant Estimates for PBJD Model (Dividend-Adjusted Daily Data)

Stock Daily	λ_u	λ_d	r_u	r_d	μ	σ
BA	0.5573	0.1618	119.23	76.32	-2.516E-3	0.0112
BS	0.8662	0.3991	66.22	62.01	-5.987E-3	0.0121
IBM	2.772E-4	0.0848	19.20	46.02	6.609E-4	0.0134

Table 7: ML Estimates for PBJD Model

Stock	λ_u	λ_d	r_u	r_d	μ	σ	ln(lkhd)	$-2\ln(\Lambda)$
BA	0.3714 (0.2349)	0.04760 (0.1431)	99.53 (1.462)	44.55 (1.015)	-2.613E-3 (1.566E4)	0.0120 (1.908E-6)	1398.43	50.69^{**}
BS	0.5198 (0.1497)	0.0232 (0.0386)	61.79 (1.684)	27.42 (0.6591)	-6.785E-3 (5.101E-4)	0.0203 (7.530E-6)	1126.88	44.08^{**}
DAL	0.3082 (0.4161)	0.06892 (0.1925)	86.29 (3.956)	73.06 (0.2468)	-2.295E-3 (5.616E-4)	0.0166 (1.201E-5)	1286.36	13.42^{**}
F	0.4224 (0.4942)	0.6437 (0.1747)	82.18 (0.9165)	142.04 (0.0383)	8.854E-4 (5.576E-4)	0.0154 (5.556E-6)	1257.57	20.44^{**}
GT	0.1076 (0.1604)	0.0416 (0.0864)	45.55 (3.905)	62.97 (0.2417)	1.277E-3 (8.449E-4)	0.0184 (1.299E-5)	1253.32	35.98^{**}
IBM	0.0884 (0.6434)	0.06307 (0.5721)	83.53 (5.874)	37.78 (2.344)	-5.669E-4 (1.553E-4)	0.0121 (4.850E-6)	1422.19	99.78^{**}

'**' ('*') indicates significances at 1% (%5) level. Standard Error in parantheses.

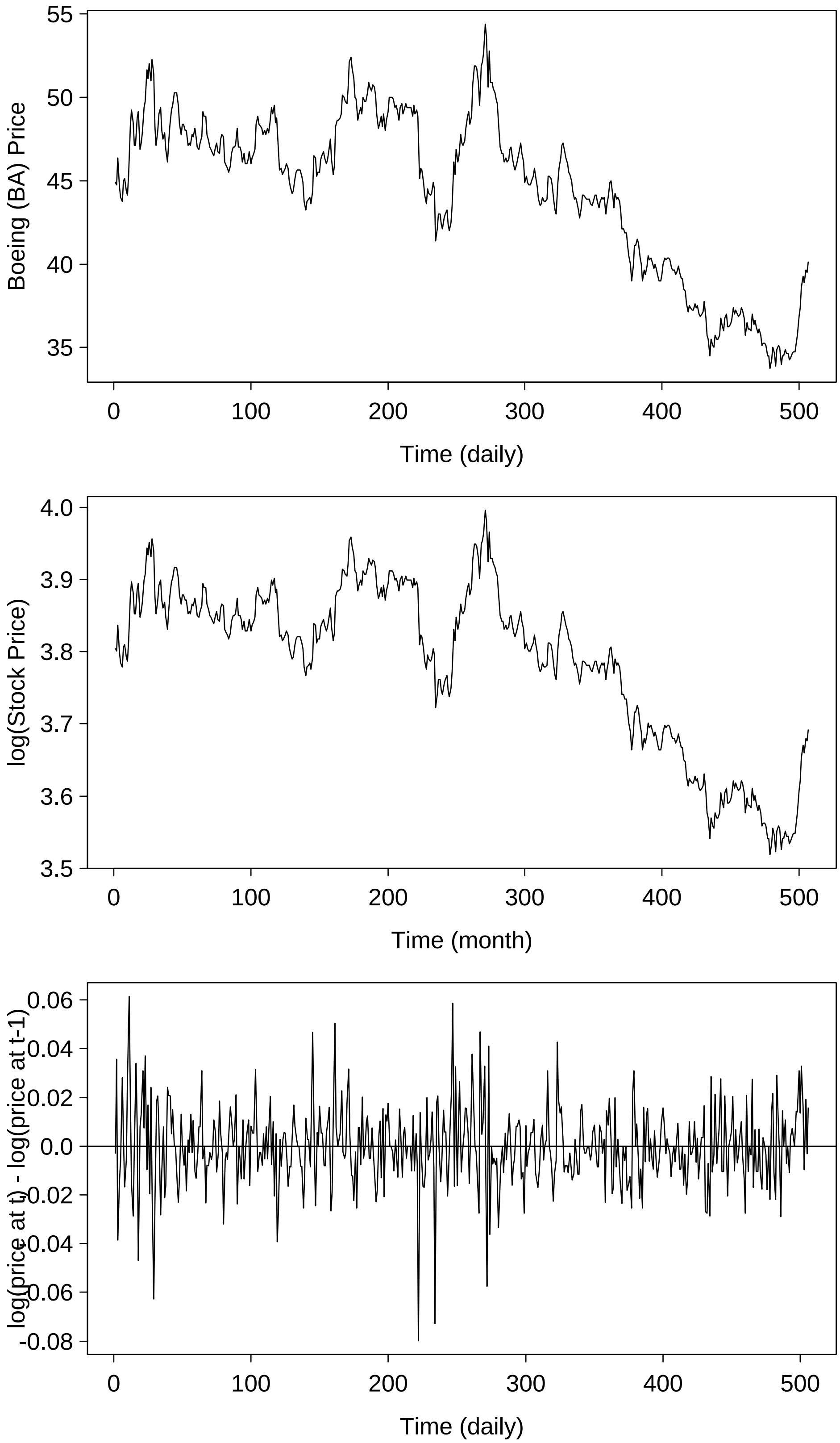

Figure 1: Time Series Plots for BA

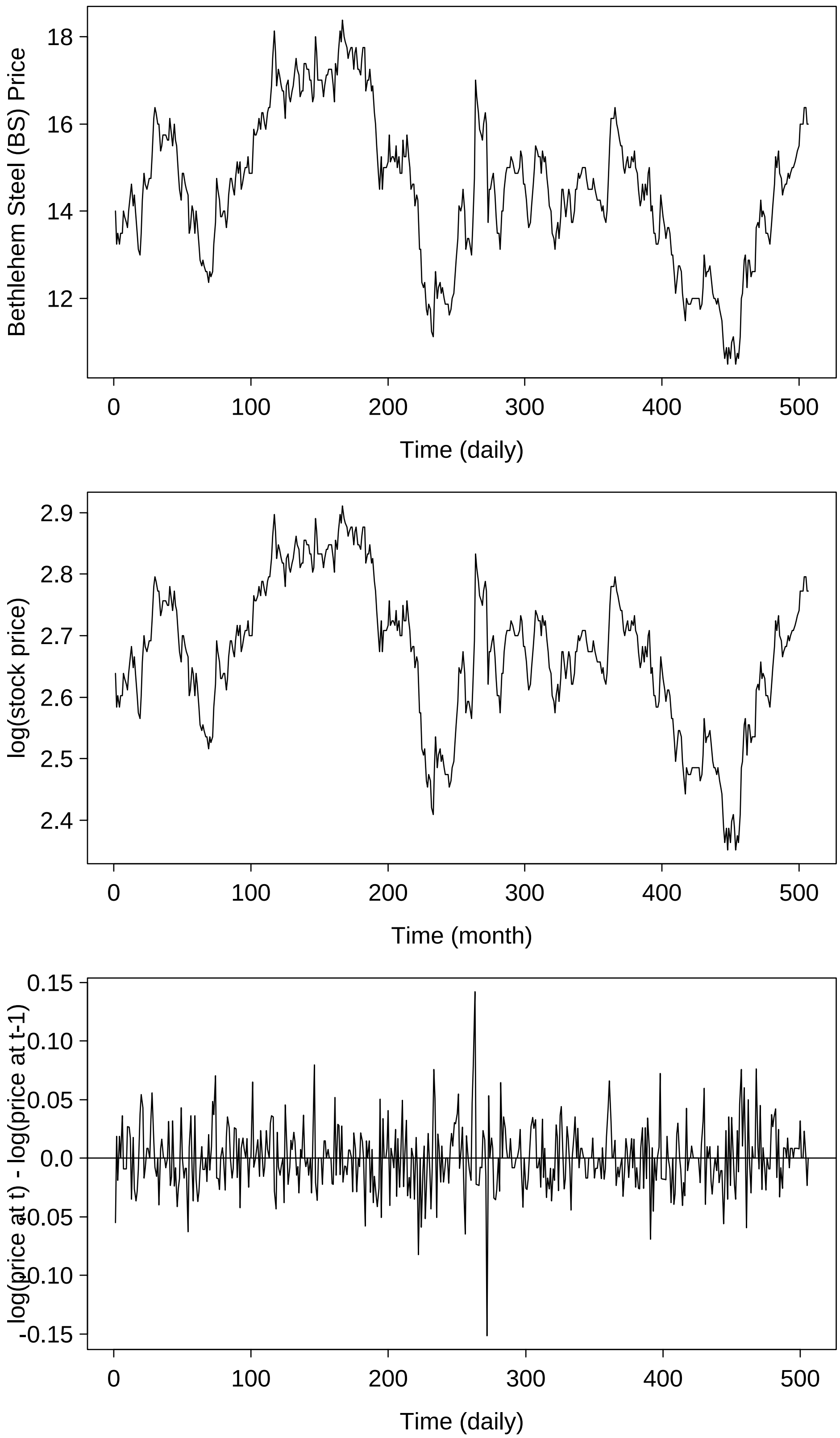

Figure 2: Time Series Plots for BS

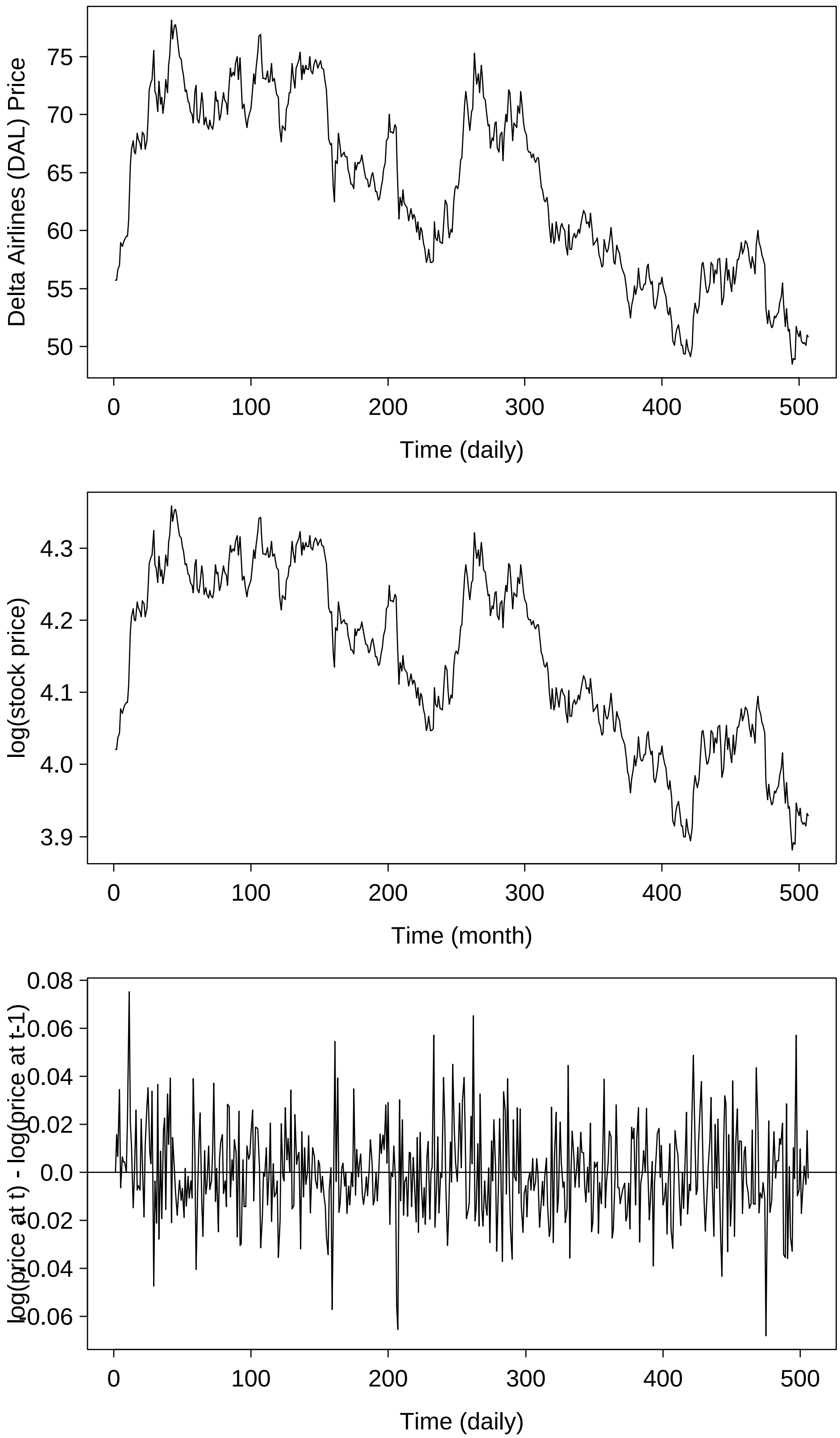

Figure 3: Time Series Plots for DAL

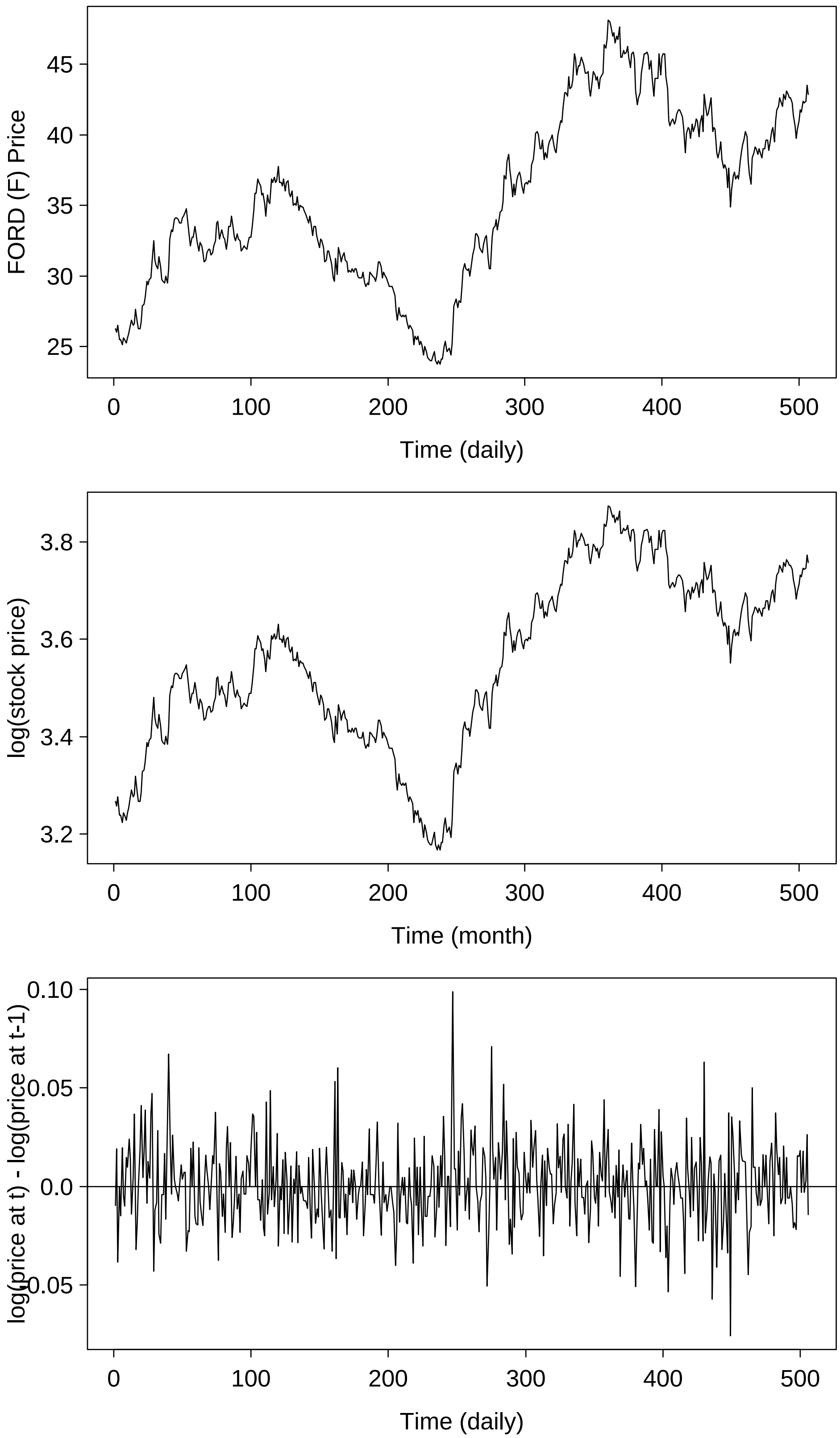

Figure 4: Time Series Plots for F

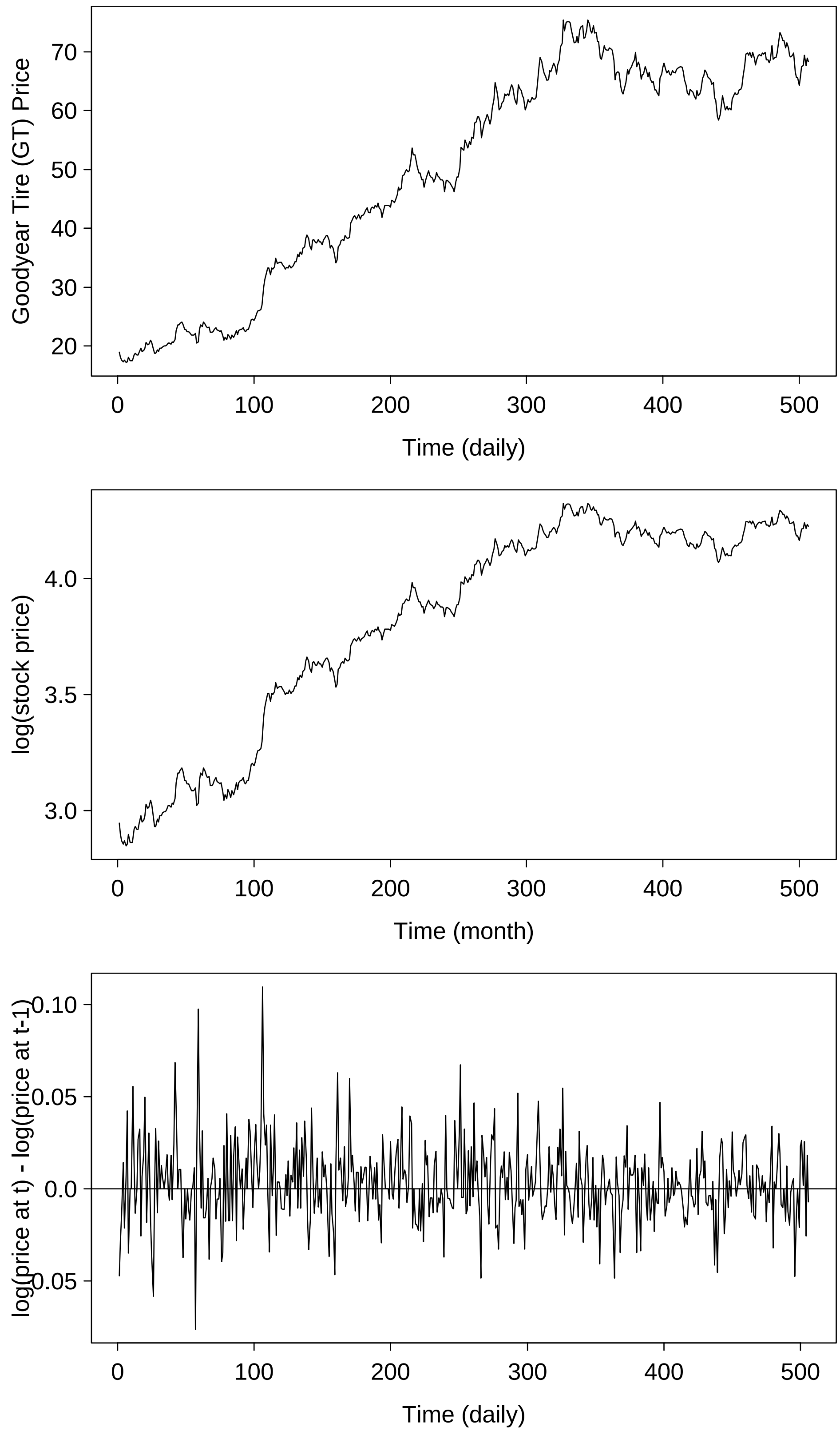

Figure 5: Time Series Plots for GT

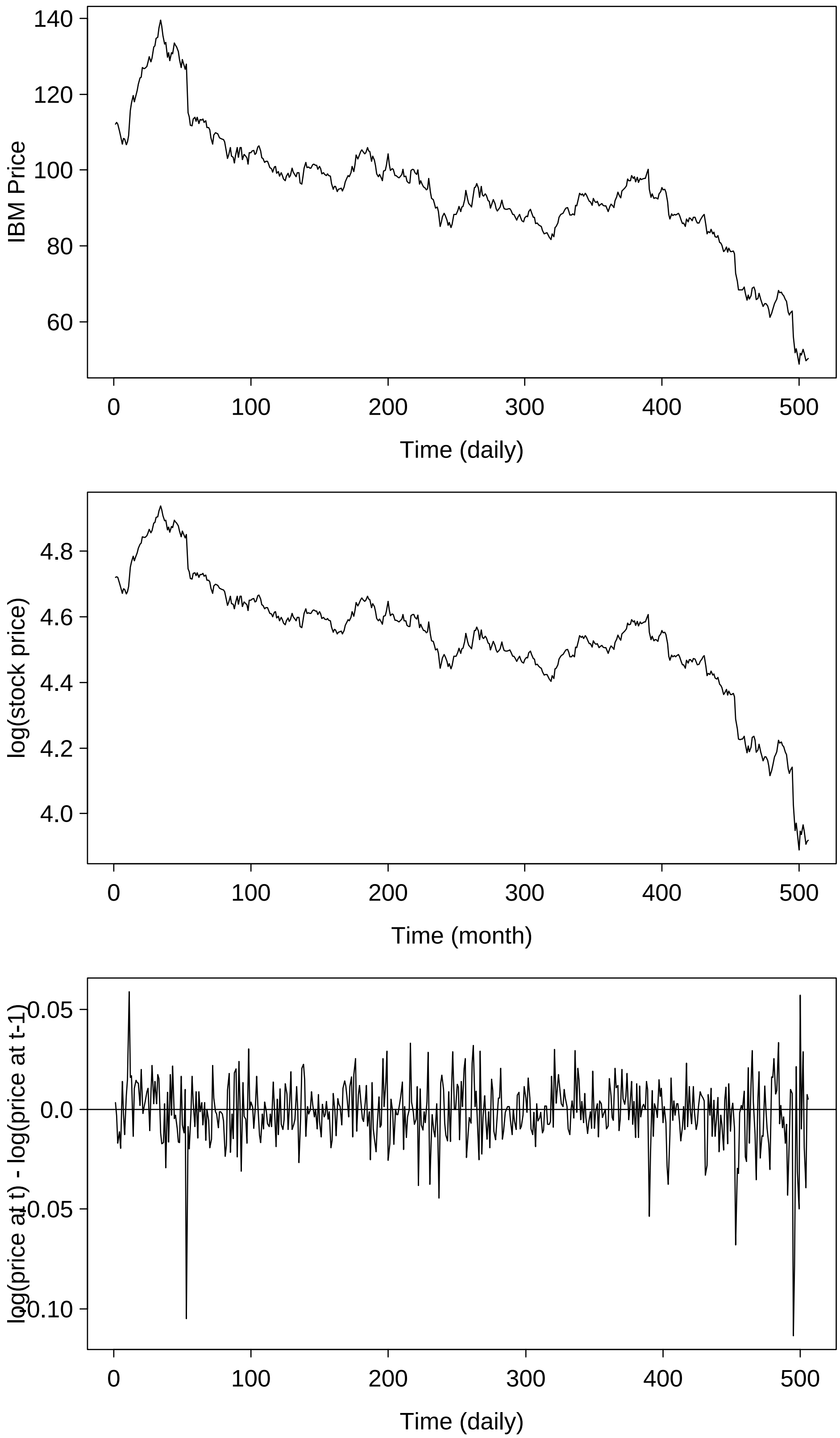

Figure 6: Time Series Plots for IBM

Notes

[1] There is a large body of empirical studies that document the existence of Leptokurtic returns distribution. For summary of recent developments see Bookstaber & McDonald (1987), Madan & Seneta (1990), and Tucker (1992) and their reference section.

[2] Tucker (1992) provides comparison of these models using daily stock returns.

[3] See Das (1998), Ho, Perraudin & Sorenson (1996), Jorion (1988), Pan (1997) and their references.

[4] For the existence of jumps in interest rates processes see Das (1997), and Das (1998). Jump in foreign exchange process is documented in Jorion (1988), Bates (1995*b*), and Bates (1995*a*).

[5] Oldfield, Rogalski & Jarrow (1977) further generalized Merton's model and proposed an autoregressive model in which jump magnitudes are autocorrelated.

[6] Note that our definition of good and bad news is specific to direction of price jump rather than a qualitative assessment of specific news item.

[7] Product cycles characterized by early market gains followed by increased competition in later phase of a products life cycle provide an additional rationale for the proposed extension.

[8] This can be proven by comparing the characteristic functions of JD and PBJD models.

[9] The last two moments are defined as: skewness $= \frac{E(X-EX)^3}{[Var(X)]^{3/2}}$ and kurtosis $= \frac{E(X-EX)^4}{[Var(X)]^2} - 3$.

[10] The GMM procedure only depends on the moments rather than the transition density. Hansen (1982) showed that under certain regularity conditions GMM estimates are consistent and asymptoticly normal. Hansen & Scheinkman (1995) considered the estimation problem for a more general class of continuous-time Markov processes. These authors have shown that under certain technical regularity conditions, GMM estimators remain consistent and asymptotically normal. The simulated moment estimators were proposed by Duffie & Singleton (1993), who provided conditions for their consistency and asymptotic normality. For recent application of the GMM methods to the problem of estimating jump-diffusion models see Ho et al. (1996) and Perraudin & Sorensen (1996) and references there in.

[11] Other studies includes Liptser & Shiryayev (1978)(Ch17), Le Breton (1976), Brown & Hewitt (1978), and Borkar & Bagchi (1982). Lo (1988) considered the MLE for generalized Itô processes with discretely (equally- or unequally-spaced) sampled data. He derived a particular functional partial differential equation which characterizes the exact likelihood function. However, he did not establish conditions for the existence of such likelihood functions. Sorensen (1991) provides conditions for the existence of likelihood function. The PBJD model satisfies the conditions presented in Sorensen (1991).

[12] Evidently this fact has been overlooked in most studies that use the maximum likelihood procedure to estimate jump-diffusion models of asset returns.

[13] See Hamilton (1994), page 139 for a complete description of Powell's optimization procedure. The computation programs we use are taken from Press et al. (1992).

[14] Strictly speaking LRT cannot be considered a formal test in the present setting because it forces the Poisson parameter to fall on the boundary of the parameter space,$\lambda_j = 0, j = u, d$ (for details see Davies (1977) and Self & Liang (1987)). However, the use of LRT to test JD versus GBM is wide spread and can be justified particularly if the reduction in the likelihood ratio test statistics is less than 2*(number of free parameters), then accepting the alternative hypothesis is equivalent to using the Akaike Information Criterion (AIC). That is LRT can only be used as an indication of the most likely model. To see this recall the definition: AIC = -2 max (likelihood) + 2 (number of free parameters).

[15] We also estimated our model using the monthly S&P-500 composite index. As expected, the jump components were not as important for monthly data. These results are available from authors upon request.

[16] These values are obtained by plugging the parameter estimates into the formulas in Table (1) and setting $s = 252$.

[17] In this case LRT has the usual interpretation since this hypothesis test is not on the boundary of the parameter space.

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