Trading in the Presence of Cointegration [1][2]

by

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Trading in the Presence of Cointegration

Abstract

The following research presents new properties of cointegrated time series that serve as a basis for a novel high frequency trading strategy. The expected profit of this strategy is always positive. Its practical implementation is illustrated using the daily closing prices of four world stock market indexes. In-sample (as well as out-of-sample) results show that the long-term dependencies of financial time series can be profitably exploited in a variant of "pairs trading" strategy. This paper includes an extended empirical study that shows the strategy's performance as a function of its parameters. The backtests presented show the daily profit and loss results for the period between 2001 to 2006. During that time the strategy significantly outperformed a simple buy-and-hold of the individual indexes.

A popular alternative investment class has become hedge funds. One of the reasons for their success is the use of a vast variety of strategies that explore different anomalies observed in the financial markets. Focus has shifted to a group of strategies that rely on “quantitative analysis” (also known as “statistical arbitrage”.) Quantitative analysis strategy developers use sophisticated statistical and optimization techniques to discover and construct new algorithms. These algorithms take advantage of the short term deviation from the “fair” securities’ prices. *Pairs trading* is one such quantitative strategy - it is a process of identifying securities that generally move together but are currently ”drifting away”. They rely on the validity of the assumption that due to the existing long-term relationship among the securities, they will eventually revert to moving together once again. Exploring such spreads has the potential to be profitable. Many equity hedge funds are using and have used similar approaches. One of the commonly used methods for finding such spreads is based on correlation. Since correlation is a short term measure, frequent rebalancing of a portfolio to reflect the constantly changing spreads has become necessary, and as a result, high transaction costs are usually observed. Additionally, there is evidence that the correlation could be a very unreliable measure during financial market crises, like the collapse of the Long Term Capital Management in 1998.

A measure of the long-term dependencies in financial time series is cointegration. Since the seminal work of Engle & Granger (1987, 1991), many researchers in finance and economics have used cointegration to model the dependencies between securities prices. Several statistical test procedures for cointegration have been developed (see Engle & Granger 1991). Tsay (2005) presented numerous examples and a detailed justification of using cointegration in modeling financial time series. Alexander et al. (2002) used cointegration to construct an index tracking portfolio. Alexander et al. (2002) showed that the optimal index tracking portfolio has stationary tracking errors, and that efficient long-short hedge strategies can be achieved with relatively few stocks and less turnover. Alexander (2001) discussed in detail all the relevant published work in finance that used different properties of cointegration for portfolio optimization or the construction of trading strategies. In summary, published research on the use of cointegration in portfolio construction (specifically on constructing time-dynamic trading strategies) is limited mainly to those aforementioned.

To better understand the relationship between cointegration and asset allocation, we need to first describe two methods for asset allocation. The process of selecting a *target asset allocation* or *index* is called *strategic asset allocation*. The variation from the target is called *tactical asset allocation*. One can think of the strategic asset allocation as a process of selecting the appropriate benchmark for a portfolio. For example, pension plans regularly go through such a benchmark selection process in order to establish their investment policies. Also, tactical asset allocation is usually identified with active portfolio management. For example, how one should maintain the allocation of sixty percent stocks and forty percent bonds over time comes into question.

Available research investigating the relationship between cointegration and asset allocation shows that cointegration affects strategic asset allocation. In other words, the decision about optimal portfolio mix (or setting the appropriate benchmark) is influenced by the common stochastic trend between assets. Lucas (1997) presented a model where a portfolio manager maximized the expected utility of total earnings over a finite time period. The associated time-series model captured cointegrating relations among the included assets. He showed that cointegration affects strategic asset allocation while error-correction mainly affects tactical asset allocation.

Using cointegration for trading or portfolio allocation implies the existence of a long-term stochastic trend. In general, this is a contradiction with the hypothesis that the stock price returns follow a random walk. Lo & MacKinley (1988) tested the random walk hypothesis with weekly stock market returns by comparing variance estimators derived from data sampled at different frequencies. If the stock returns followed a random walk, then the variance should have grown with the square root of time. Lo & MacKinley (1988) rejected the random walk hypothesis and showed that the rejection was due largely to the behavior of small stocks. Additionally, they showed that the autocorrelations of individual securities were generally negative, and the autocorrelations of equally and positively weighted CRSP indexes were positive.

Technical trading strategies that explore different short term market inefficiencies have also been widely used by hedge fund managers. Gatev et al. (2006) showed how to construct a "pairs trading" strategy with profits that typically exceeded conservative transaction-cost estimates. Gatev et al. (2006) linked the profitability of this strategy to the presence of a common factor in the returns different from conventional risk measures.

Lo & MacKinley (1990) showed a particular contrarian strategy that sold "winners" and bought "losers" with a positive expected return, and that despite negative autocorrelations in individual stock returns, weekly portfolio returns were strongly and positively autocorrelated due to important cross-autocorrelations. Brown & Jennings (1989) showed that technical analysis (or use of past prices to infer private information) had value in a model where prices were not fully revealing, and where traders had rational conjectures about the relationship of prices to signals. Conrad & Kaul (1998) analyzed a wide range of trading strategies from 1926-1989. They showed that momentum and contrarian strategies were equally likely to be successful. Additionally, they found that the cross–sectional variation in mean-returns of individual securities included in the strategies was an important determinant of their profitability. Conrad & Kaul (1998) stated that the cross-sectional variation could potentially account for the profitability of momentum strategies, and it could also be responsible for some of the profits from price reversals to long-horizon contrarian strategies. Cooper (1999) showed evidence of predictability by filtering lagged returns and lagged volume information to uncover weekly overreaction profits on large cap stocks.

Our strategy evolves from the above described history of models and strategies. We believe that our theoretical and empirical results improve the understanding of how financial markets adjust to new information. Our trading strategy exploits short-term pricing anomalies revealed through deviation from the long-term stochastic trend. These anomalies are consistent with the weak form of cross-market integration as documented in Chen & Knez (1995). Our results also show support for the cointegrated type of asset pricing models as in Bossaerts (1988).

Throughout this paper we derive new properties of cointegrated time series. These new properties help to construct a dynamic trading strategy. It is proved that the expected return of the strategy is positive. Its practical implementation is reached using four world stock market indexes, and presented is a detailed analysis of its performance. In-sample (as well as out-of-sample) results show that long-term dependencies can be profitably exploited in a variant of "pairs trading" strategy. The results do not vanish when we perform an out-of-sample test. The problems with in-sample overfitting were well documented by Bossaerts & Hillion (1999), Pesaran & Timmermann (1995), Cooper (1999), and Conrad et al. (2003). Additionally, the "data snooping" problem and its relation to out-of-sample tests (as documented by Conrad et al. 2003, Cooper & Gulen 2006, Sullivan et al. 1999) is not a problem for us, since our trading strategy is based on a theoretical relationship that we identify for cointegrated time series. In particular, we prove that cointegration is a property related to the 1st and 2nd moments of asset returns. In previous work the cointegration was viewed as a property of the asset prices. We show that under certain assumptions the cointegration is defined by the stochastic relationships among the asset returns.

This paper is organized as follows. First, the theoretical results are presented in §1. Then, the strategy construction and the proof of its positive expected return are given in §2. Finally, the detailed in-sample and out-of-sample empirical studies are in §3.

1 Stochastic Dependencies in Financial Time Series

Portfolio allocation models, like mean-variance (introduced by Markowitz (1952)) and Value-At-Risk (first introduced by Jorion (2001)), included a measure of the risk associated with the set of assets being considered. The standard measure included is the corresponding covariance (or equivalently - the correlation) matrix. The covariance between two random vectors X and Y is given by

$$Cov(X, Y) = E\left\{(X - E[X])\,(Y - E[Y])^T\right\},$$

where $E[X]$ stands for the expected value of the random vector X, and T denotes the matrix transpose operator. It is well documented that the covariance (or correlation) is a measure of the short-term linear dependencies (see Casella & Berger 2002, Theorem 4.5.7).

In contrast to correlation, cointegration is a measure of long-term dependencies (see Engle & Granger 1991). We will briefly outline the notion of cointegration. In order to do so, we need to first define stationary and integrated time series.

A stochastic process Y_t is stationary if its first and second moments are time invariant: in particular if $E[Y_t] = \mu$, $\forall t$ and $E[(Y_t - \mu)(Y_{t-h} - \mu)^T] = \Gamma_Y(h) = \Gamma_Y(-h)^T$, $\forall t, h = 0, 1, 2, \ldots$, where μ is a vector of finite mean terms, and $\Gamma_Y(h)$ is a matrix of finite covariances. Such a process is known as *integrated of order 0* and denoted by $I(0)$. Univariate process is called *integrated of order d*, $I(d)$, if in its original form it is non-stationary but becomes stationary after differencing d times. If all elements of the vector X_t for $t = 1, \ldots, N$, are $I(1)$, and there exists a vector b such that $b^T X_t$ is $I(0)$, then the vector process X_t is said to be cointegrated, and b is called the cointegrating vector. For example, two time series X and Y are cointegrated if X, Y are $I(1)$, and there exists a scalar b such that $Z = X - bY$ is $I(0)$.

Assume now that we have $N \geq 2$ cointegrated financial assets, and their log-prices are $I(1)$ processes. It is widely assumed that stock returns are integrated of order 0, whereas the stock prices are integrated of order 1 (see Alexander et al. 2002).

Denote the vector of the asset prices by $P_t = \{P_t^1, \ldots, P_t^N\}$. Each of its elements can be written as $P_t^i = P_0^i e^{\sum_{j=0}^t r_j^i}$, $i = 1, \ldots N$, where $r = \{r_t^1, \ldots, r_t^N\}$ are the continuously compounded asset returns, and $P_0^1, \ldots, P_0^N$ are the initial prices. (Without loss of generality, we can assume that $P_0^1 = \ldots = P_0^N = 1$). Then, the log-prices can be written as $\ln P_t^i = \ln P_0^i + \sum_{j=0}^t r_j^i, i = 1, \ldots, N$. Denote the corresponding cointegrating vector by $b = (b^1, \ldots, b^N)$. By the definition of cointegration, the resulting time series $Y_t = \sum_{i=1}^N b^i \ln P_t^i$ will be stationary and integrated of order 0.

The next two propositions lead to derivations of new properties of cointegrated time series that we later use for the construction of a new trading strategy.

Proposition 1 *Assume that the log prices of* $N, N \geq 2$ *assets,* $\ln P^i, i = 1, \ldots, N$*, are cointegrated with a cointegrating vector b. Let* $Y_t = \sum_{i=1}^N b^i \ln P_t^i$ *be the corresponding stationary series, and* $\{r_t^1, \ldots, r_t^N\}$ *be the continuously compounded asset returns at time* $t > 0$*. Define* $Z_t = Y_t - Y_{t-1} = \sum_{i=1}^N b^i r_t^i$*. If* $\lim_{p \to \infty} Cov[Y_t, Y_{t-p}] = 0$*, then* $\sum_{p=1}^{\infty} p Cov[Z_t, Z_{t-p}] = -Var Y_t$*, where* $Cov[Z_t, Z_{t-p}] = \sum_{i=1}^N \sum_{j=1}^N b^i b^j Cov[r_t^i, r_{t-p}^j]$.

Proof: See the Appendix.

Proposition 1 is a technical result that we need to prove the next proposition. Intuitively, it shows that the variance of the cointegration process (Y_t) inadvertently defines the auto-covariance of the asset returns.

Proposition 2 *Assume* $\ln P_t^i, i = 1, \ldots, N$ *are the log-prices of* N *assets, and* $\{r_t^1, \ldots, r_t^N\}$ *are the continuously compounded asset returns at time* $t > 0$. *For some finite vector* b*, the process* $Y_t = \sum_{i=1}^N b^i \ln P_t^i$*, given* $\lim_{p\to\infty} Cov[Y_t, Y_{t-p}] = 0$*, is stationary, and therefore the time series of the assets' log-prices are cointegrated,* **if and only if** *the process* $Z_t = \sum_{i=1}^N b^i r_t^i$ *has the following three properties:*

(i) $EZ_t = 0$

(ii) $VarZ_t = -2\sum_{p=1}^\infty Cov[Z_t, Z_{t-p}]$

(iii) $\sum_{p=1}^\infty pCov[Z_t, Z_{t-p}] < \infty$.

Proof: See the Appendix.

As a result of proposition 2, it follows that cointegration is a property related to the 1st and 2nd moments of asset returns. In previous work, cointegration was viewed as a property of asset prices. Here we show that under certain assumptions, cointegration is defined by the stochastic relationships among the returns.

Now presented is an example of cointegrated time series which illustrates the results of the two propositions. Consider a model describing the log-return movements of two assets r_t^1 and r_t^2:

$$\begin{aligned} r_t^1 &= r_t^2 + c + (\gamma - 1)Y_{t-1} + \xi_t^1 \\ r_t^2 &= \mu_2 + \xi_t^2 \\ Y_t &= c + \gamma Y_{t-1} + \xi_t^1, \end{aligned}$$

where μ_2 is a real valued constant, and (ξ_t^1, ξ_t^2) are bivariate normal random variables with mean vector 0, variances σ_1^2, σ_2^2, and covariance ρ. γ is a parameter of the AR(1) process Y_t, and it is a constant between 0 and 1, whereas c is a parameter of the AR(1) process Y_t, a constant.

The log-prices of these two assets are cointegrated of order one with cointegration vector $(1, -1)$. The corresponding theoretical moments are:

$$
\begin{aligned}
E[r_t^1] &= \mu_2, E[r_t^2] = \mu_2 \\
Var[r_t^1] &= \sigma_2^2 + 2\rho + 2\sigma_1^2/(1+\gamma), Var[r_t^2] = \sigma_2^2 \\
Cov[r_t^1, r_t^2] &= \sigma_2^2 + \rho, Cov[r_t^1, r_{t-p}^2] = (\gamma - 1)\rho\gamma^{p-1}, p = 1, 2, \dots \\
Cov[r_t^1, r_{t-p}^2] &= -(1-\gamma)\sigma_1^2\gamma^{p-1}/(1+\gamma) + (\gamma-1)\rho\gamma^{p-1}, p = 1, 2, \dots \\
Cov[r_t^2, r_{t-p}^1] &= 0, \quad p = 1, 2, \dots, Cov[r_t^2, r_{t-p}^2] = 0, p = 1, 2, \dots.
\end{aligned}
$$

We want to show that both propositions for this model hold. Below are the left- and the right-hand sides of the propositions' equations. The equality holds for every pair.

$$
\begin{aligned}
E[Z_t] &= E[Y_t - Y_{t-1}] = 0 \\
Var[Z_t] &= Var[c + (\gamma - 1)Y_{t-1} + \xi_t^1] = \frac{2\sigma_1^2}{(1+\gamma)} \\
Var[Y_t] &= \frac{\sigma^2}{1-\gamma^2} \\
Cov[Z_t, Z_{t-p}] &= \text{Cov}[r_t^1, r_{t-p}^1] + \text{Cov}[r_t^2, r_{t-p}^2] - \text{Cov}[r_t^1, r_{t-p}^2] - \text{Cov}[r_t^2, r_{t-p}^1] \\
&= \frac{(1-\gamma)\sigma_1^2}{1+\gamma}\gamma^{p-1} \\
\sum_{p=1}^{\infty} \text{Cov}[Z_t, Z_{t-p}] &= -\frac{(1-\gamma)\sigma_1^2}{(1+\gamma)} \sum_{p=0}^{\infty} \gamma^p = -\frac{\sigma_1^2}{(1+\gamma)} \\
\sum_{p=1}^{\infty} p\text{Cov}[Z_t, Z_{t-p}] &= -\frac{(1-\gamma)\sigma_1^2}{(1+\gamma)\gamma} \sum_{p=0}^{\infty} p\gamma^p = -\frac{\sigma_1^2}{1-\gamma^2}
\end{aligned}
$$

2 Construction of a Daily Trading Strategy

Next, we introduce a trading strategy by exploiting the theoretical results derived in the previous section.

Summarizing the results from the two propositions: for process $Z_t = Y_t - Y_{t-1} = \sum_{i=1}^{N} b^i r_t^i$, we have that $Var Z_t = -2\sum_{p=1}^{\infty} Cov[Z_t, Z_{t-p}]$ and $EZ_t = E[Y_t - Y_{t-1}] = 0$. Consider a strategy where each time period we buy $-b^i C \sum_{p=1}^{\infty} Z_{t-p}$ value of stock i, $i = 1, \dots, N$ and sell everything in the next time period. C is a positive scale factor. The reason for which we include constant C will become clear later.

At any point in time we can compute the profit of this strategy by multiplying the next period return by the shares purchased:

$$\pi_t = \sum_{i=1}^{N} -b^i C \left[\sum_{p=1}^{\infty} Z_{t-p} \right] r_t^i = -C \sum_{p=1}^{\infty} Z_{t-p} Z_t.$$

Given that $EZ_t = 0$ and $Cov[Z_t, Z_{t-p}] = EZ_t Z_{t-p} - EZ_t Z_{t-p}$,$p > 0$, the expected profit of this strategy is:

$$\begin{aligned} E[\pi_t] &= E\left[-C \sum_{p=1}^{\infty} Z_{t-p} Z_t \right] = -C \sum_{p=1}^{\infty} Cov\left[Z_t, Z_{t-p}\right] = \\ &= 0.5 C Var Z_t. \end{aligned}$$

Since $VarZ_t$ and C are positive, the expected profit of the proposed strategy is always positive and proportional to the scale factor C.

The reasoning behind this strategy is fairly simple. The cointegration relations between time series imply that the time series are bound together. Over time the time series might drift apart for a short period of time, but they ought to re-converge. The term $\sum_{p=1}^{\infty} Z_{t-p}$ measures how far they diverge, and $sign\left(-b^i \sum_{p=1}^{\infty} Z_{t-p}\right)$ (where $sign(a) = +1$ if $a > 0$ and -1 if $a < 0$) provides the direction of the trade for stock i. Specifically, $+1$ stands for a long position, whereas -1 denotes a short trade. Such a trading strategy is a variant of pairs-trading where one usually bets on identifying spreads that have gone apart but are expected to mean revert in the future. The spreads of typical pairs-trading strategy get identified by using correlation as a similarity measure and standard deviation as a spread measure. A trade, for example, will be put in place if the assets are highly correlated but have gone apart for more than 3 standard deviations. The trade will unwind when the assets converge or some time limit is reached.

Our approach uses cointegration as a measure of similarity. Cointegration is the natural answer of the question: How do we identify assets that move together? Proposition 2 provides the answer of the question: How far do the assets have to diverge before a trade is placed? As a result, the decision to execute a trade is driven by cointegration properties of the assets.

Having positive expected profit is excellent news for any strategy. The proposed strategy has some shortcomings. The initial amount of money needed each period is a random variable, and the resulting portfolio is not dollar neutral (i.e. the total dollar value of the long position is not equal to the total dollar value of the short position.) To construct a dollar neutral long-short portfolio, we will first partition the cointegrated time series into two sets L and S:

$$i \in L \leftrightarrow b^i \geq 0$$
$$i \in S \leftrightarrow b^i < 0.$$

Next, depending on what set a given asset belongs to, we purchase the value of

$$-b^i C sign\left(\sum_{p=1}^{\infty} Z_{t-p+1}\right) / \sum_{j \in L} b^j, \quad i \in L$$
$$b^i C sign\left(\sum_{p=1}^{\infty} Z_{t-p+1}\right) / \sum_{j \in S} b^j, \quad i \in S.$$

The return of this modified strategy is identical to the proposed earlier. Hence, the expected profit for that strategy is also positive.

Indeed (without loss of generality) assume that $sign\left(\sum_{p=1}^{\infty} Z_{t-p}\right) = -1$. The long returns R_t^L and the short returns R_t^S of our original strategy are

$$\begin{aligned} R_t^L &= \frac{\sum_{i \in L} -b^i C \sum_{p=1}^{\infty} Z_{t-p} P_{t+1}^i / P_t^i}{\sum_{i \in L} -b^i C \sum_{p=1}^{\infty} Z_{t-p}} - 1 = \frac{\sum_{i \in L} b^i P_{t+1}^i / P_t^i}{\sum_{i \in L} b^i} - 1 \\ R_t^S &= 1 - \frac{\sum_{i \in S} -b^i C \sum_{p=1}^{\infty} Z_{t-p} P_{t+1}^i / P_t^i}{\sum_{i \in S} -b^i C \sum_{p=1}^{\infty} Z_{t-p}} = 1 - \frac{\sum_{i \in S} b^i P_{t+1}^i / P_t^i}{\sum_{i \in S} b^i}, \end{aligned}$$

where sets S and L are defined above.

The modified strategy has the following returns from the short and long positions:

$$\begin{aligned} R_t^L &= \frac{\sum_{i \in L} \frac{-b^i}{\sum_{j \in L} b^j} C sign(\sum_{p=1}^{\infty} Z_{t-p}) P_{t+1}^i / P_t^i}{\sum_{i \in L} \frac{-b^i}{\sum_{j \in L} b^j} C sign(\sum_{p=1}^{\infty} Z_{t-p})} - 1 = \frac{\sum_{i \in L} b^i P_{t+1}^i / P_t^i}{\sum_{i \in L} b^i} - 1 \\ R_t^S &= 1 - \frac{\sum_{i \in S} \frac{b^i}{\sum_{j \in S} b^j} C sign(\sum_{p=1}^{\infty} Z_{t-p}) P_{t+1}^i / P_t^i}{\sum_{i \in S} \frac{b^i}{\sum_{j \in S} b^j} C sign(\sum_{p=1}^{\infty} Z_{t-p})} = 1 - \frac{\sum_{i \in S} b^i P_{t+1}^i / P_t^i}{\sum_{i \in S} b^i}. \end{aligned}$$

The above derivations indicate the return of the modified strategy is the same as the original one, therefore its expected profit is positive(since we proved that the expected return of the original strategy is positive).

Now we can explain why we have included the constant C. In the modified strategy, every time period the value of C is invested in short and long positions. Hence, the money needed for each time period in order to execute the new strategy is a constant, and the portfolio we obtain is dollar neutral.

In reality, we cannot compute the true value of $\sum_{p=1}^{\infty} Z_{t-p}$ (the cointegration vector b.) We estimate them, and with the above theoretical results in mind, we propose the following trading strategy:

- Step 1: using historical data, estimate the cointegration vector b.
- Step 2: using the estimated cointegration vector $\tilde{b}$ and historical data, construct $\tilde{Z}_t$ - realizations of the process $Z_t = \sum_{i=1}^{N} b^i r_t^i$.
- Step 3: compute the final sum $\sum_{p=1}^{P} \tilde{Z}_{t-p+1}$, where P is a parameter.
- Step 4: partition the assets into two sets L and S (depending on values of $\tilde{b}$.)
- Step 5: buy (depending in which set the asset belongs to) the following number of shares (round down to get integer number of shares):

$$
\begin{aligned}
-\tilde{b}^i C sign\left(\sum_{p=1}^{P} \tilde{Z}_{t-p+1}\right) / \left[P_t^i \sum_{j\in L} \tilde{b}^j\right], \quad i \in L \\
\tilde{b}^i C sign\left(\sum_{p=1}^{P} \tilde{Z}_{t-p+1}\right) / \left[P_t^i \sum_{j\in S} \tilde{b}^j\right], \quad i \in S.
\end{aligned}
$$

- Step 6: close all the open positions the following trading day.
- Step 7: update the historical data set.
- Step 8: if it is time to re-estimate the cointegration vector (which happens every 22 trading days), go to step 1, otherwise go to step 2.

In the next section we describe the procedures used to test the strategy and present the numerical results.

3 Trading Strategy - Performance and Analysis

To test the proposed trading strategy, we use historical data for four equity world indexes: AEX[4], DAX [5], CAC[6] and FTSE[7]. Then, we present the results from several backtests. The process of backtesting is an implementation of a trading strategy on available historical data where *we go back in time* and start trading while pretending the past is now. It gives us a picture of how the strategy would have performed if we were to use it in the past. Such procedures are commonly used by financial professionals when a particular trading rule/strategy is suggested for actual trading.

There are two types of tests: in-sample and out-of-sample. For in-sample backtesting, all available data is used to estimate all the parameters of the strategy. Performance measures are always better for in-sample testing, since we pretend that we can "see" the future by using all the available historical information. Out-of-sample testing is a fair backtesting process where information is available only up to the trading moment.

The number of days used to perform the estimation of the parameters is called a window and is denoted by W. For example, a window size of 1000 ($W = 1000$) means that we used the last 1000 daily observations for estimation.

Realizations from 01/02/1996 to 12/28/2006 were used for AEX, DAX, CAC and FTSE indexes. Missing data due to difference in working days in different countries was filled using one of the SPLUS FinMetrics functions that interpolates the missing value using a spline. We also used the SPLUS vector auto regression estimation function to get the order of the autoregressive process and its Johansen (1995) rank test to estimate the cointegration vector. Trading for all tests starts on 11/06/2001 and ends on 12/28/2006. Transaction cost per share was set to 1 cent. For out-of-sample testing we re-estimated the cointegration vector b every 22 days. The value of the long/short positions each day was set to 10,000,000 dollars (this value may differ due to rounding.)

3.1 In-sample Results

To estimate the parameters for the in-sample test, we used all available historical data. Results from this test for different values of P (the lag parameter) can be found in Tables 1 and 2, where Table 1 shows the results without transaction costs, and while Table 2 shows the statistics after transaction costs of 1 cent per share. The long-term relationship between the four indexes is estimated and described by the following cointegration vector:

$$Z = 3.69 \times AEX - 4.66 \times CAC + 13.57 \times DAX - 21.49 \times FTSE.$$

The resulting process Z is $I(0)$.

Insert Tables 1 and 2 here.

Both tables report various performance measures: best and worst days, percentage of up and down days, average daily gains and losses, volatility of positive and negative returns, Sharpe and Sortino ratios, median, skewness and kurtosis of the daily returns, as well as the average, standard deviation and maximum run down[8].

As the value of the lag parameter changes from 10 to 40 days, the performance statistics change as well. The best results (in terms of the corresponding Sharpe and Sortino ratios) are for $P = 30$ days. For that particular value of the lag parameter, the Sharpe ratio is 1.15 and the Sortino ratio is 1.66. The annual return is 15.43% with a volatility of 13.44%. The maximum number of days with consecutive negative returns is 8, and the average run down days are 2. The performance statistics are based on 1320 trading days. The total return for the covered period is 80.82% without transaction costs and 78.23% with transaction costs.

Table 2 shows in-sample results after transaction costs. Note that the results slightly deteriorate, but the pattern stays the same. For that reason, we will report results without transaction costs, since they are different for different types of investors[9].

Figure 1 shows the profit and loss plot (P&L) for the strategy, as well as the four market indexes. The trading strategy performs better than a simple buy-and-hold of the individual indexes.

Insert Figure 1 here.

Table 3 shows the correlation between the strategy's and individual indexes' daily returns. The results show that the performance of strategy is almost neutral or negatively correlated with the individual indexes.

Insert Table 3 here.

3.2 Out-of-sample Results

An out-of-sample test is the only available tool to truly test how a strategy would have performed if traded in the past. One of its drawbacks is the fact that we are testing the performance of the strategy on one sample path only. Another drawback is that we face the Uncertainty Principle of Heisenberg (see Bernstein 2007). The practical interpretation is that one cannot measure the risk of an asset on the basis of past data alone. Once we enter the first trade, effectively we are changing the market, and any type of backtest will not be a perfect predictor about future performance.

We designed several out-of-sample backtests by changing the values of the critical parameters. All of them start on 11/06/2001. The amount of historical data used (or the size of the window) varies from 1000 to 1500 days. For the first set of out-of-sample tests, we used a sliding window to compute the parameters. Every 22 days the parameters of the cointegration vector were re-estimated by using the last 1000, 1250 and 1500 days. A second set of out-of-sample backtests used a cumulative window, where the initial window was 1000, 1250 and 1500 days. With every new trading day, the historical window used increased by one day.

Tables 4, 5 and 6 show performance measures for the first set of out-of-sample tests with sliding windows 1000, 1250 and 1500, respectively.

Insert Tables 4, 5 and 6 here.

Note that relative to the in-sample results, the strategy performance did not change dramatically. For values of the lag parameter equal to 25 or 30 days, the Sharpe ratio varied between 0.93 and 1.51, while the Sortino ratio varied between 1.26 and 2.26. The best results in terms of Sharpe and Sortino ratio were obtained for a sliding window of 1000 days and a lag parameter of 25 days. For this particular case the Sharpe ratio was 1.35 and the Sortino ratio was 1.98.

Figure 2 presents the total return of the strategy (fixed window setup) as a function of the two parameters of interest: the lag P and the window size W. Figure 3 is the same for the cumulative window case. From here we can conclude that the lag parameter is more important for the performance of the strategy than the window size.

Insert Figures 2 and 3 here.

Figure 4 shows the profit and loss for the strategy, when the lag parameter is 25 days and the sliding window is 1000 days. Similar to the in-sample test, the strategy performs better than buy-and-hold of the four indexes.

Insert Figure 4 here.

Table 7 shows the correlations of the strategy with the individual indexes. Similar to the in-sample test, the strategy is almost neutral to the four market indexes.

Insert Table 7 here.

Figure 5 shows the time series of the cointegration vector.

Insert Figure 5 here.

Figures 4 and 5, if analyzed together, show an interesting property of the long-term relationship between the indexes. The values of the cointegration vector do not change substantially from 11/06/2001 to 3/25/2003. During this period, the strategy performs extremely well. During the same time period, all four indexes lose approximately 60% of their value, while the strategy gains almost 50%. The cointegration vector changes dramatically on 3/25/2003 - a couple of days after the start of the Iraq war[10]. The next time point at which a major adjustment in the parameters of the cointegration vector occurred is 8/16/2005, almost one month after the terrorist attack in London.

Tables 8, 9 and 10 show performance measures for the second set of out-of-sample tests. Cumulative windows are used starting with 1000, 1250 and 1500 days, respectively. The best results in terms of Sharpe and Sortino ratio were obtained for lag parameter of 25 days and cumulative window starting at 1500 days. The Sharpe ratio was 1.39, and the Sortino ratio was 2.02.

Insert Tables 8, 9 and 10 here.

Figure 6 shows the profit and loss for the strategy when the lag parameter is 25 days, and the cumulative window starts at 1500 days. Similar to the in-sample test, the strategy performs better than buy-and-hold of the four indexes.

Insert Figure 6 here.

Table 11 shows the correlations of the strategy with the individual indexes. Similar to the in-sample test, the strategy is almost neutral to the four market indexes.

Insert Table 11 here.

Figure 7 shows the time series of the cointegration vector.

Insert Figure 7 here.

With a cumulative window, the values of the cointegration vector are not highly variable. The only period in time when some changes are visible is at the start of the Iraq war. One can conclude from the two different designs of out-of-sample tests that the Iraq war caused some major changes in the long-term relationship among financial assets.

Performance statistics from Tables 4 and 10 indicate that sliding window design is better than cumulative window design. This outcome is not surprising if we refer to Proposition 1. One of the assumptions there is that there will be a value of the lag parameter P for which the long-term relationship will be almost non-existent. The results from the sliding window design are empirical confirmation for that assumption.

4 Conclusion

We have derived a set of new properties of cointegrated financial time series. Using them allowed us to create a new trading strategy. We proved that the expected profit of this strategy is always positive, and we showed its practical implementation by using the daily closing prices of four world stock market indexes.

In-sample and out-of-sample tests showed that the designed strategy significantly outperformed a simple buy-and-hold of the individual indexes. Additionally, the time series of the cointegration vector exhibited how the strategy adapts to big stress events (like the start of the Iraq war and the terrorist attack in London) in the financial markets.

5 Figure Legends

Figure 1: This figure shows the dynamics of the cumulative returns from the out-of-sample backtest done with a lag of 25 days and a window size of 1000 days without transaction costs, compared to the corresponding returns of the four indexes (AEX, CAC, DAX, FTSE.) The horizontal axis is the time, and the vertical axis is the total cumulative return.

Figure 2: This figure is a 3 dimensional plot of the total return of the cointegration daily strategy (out-of-sample test) as a function of the window size and the lag. The strategy was run using the following values of the window size 1000, 1250 and 1500; the values of the lag were 10,15,20,25,30,35 and 40.

Figure 3: This figure is a 3 dimensional plot of the total return of the cointegration daily strategy (out-of-sample aggregate test) as a function of the window size and the lag. The strategy was run using the following values of the window size 1000, 1250 and 1500; the values of the lag were 10,15,20,25,30,35 and 40.

Figure 4: This figure shows the dynamics of the cumulative returns from the out-of-sample backtest done with a lag of 25 days and a window size of 1000 days without transaction costs, compared to the corresponding returns of the four indexes (AEX, CAC, DAX, FTSE). The horizontal axis is the time, and the vertical axis is the total cumulative return..

Figure 5: This figure shows the estimated cointegration vector as a function of time. The vector is re-estimated every 22 days using the Johansen cointegration rank test with data from the previous 1000 days.

Figure 6: This figure shows the dynamics of the cumulative returns from the out-of-sample backtest done with a lag of 25 days and a cumulative window starting at 1500 days without transaction costs, compared to the corresponding returns of the four indexes (AEX, CAC, DAX, FTSE.) The horizontal axis is the time, and the vertical axis is the total cumulative return.

Figure 7: This figure shows the estimated cointegration vector as a function of time. The vector is re-estimated every 22 days using the Johansen cointegration rank test using all data up to the current time period.

6 Appendix

Proof of Proposition 1:

Define $Z_t = Y_t - Y_{t-1} = \sum_{i=1}^{N} b^i(\ln P_t^i - \ln P_{t-1}^i) = \sum_{i=1}^{N} b^i r_t^i$, where $r_t^i = \ln P_t^i - \ln P_{t-1}^i$. We can write the covariance between Y_t and Y_{t+p} for some lag $p > 0$ as:

$$
\begin{aligned}
Cov[Y_t, Y_{t+p}] &= Cov[Y_t, Y_t + Y_{t+1} - Y_t + Y_{t+2} - Y_{t+1} + \ldots + Y_{t+p} - Y_{t+p-1}] \\
&= Cov[Y_t, Y_t + \sum_{j=1}^{p} Z_{t+j}] \\
&= VarY_t + \sum_{j=1}^{p} Cov[Y_t, Z_{t+j}] \\
&= VarY_t + \sum_{j=1}^{p} \sum_{l=-\infty}^{t} Cov[Z_l, Z_{t+j}] \\
&= VarY_t + Cov[Z_t, Z_{t+p}] + Cov[Z_{t-1}, Z_{t+p}] + Cov[Z_{t-2}, Z_{t+p}] + \ldots \\
&+ Cov[Z_t, Z_{t+p-1}] + Cov[Z_{t-1}, Z_{t+p-1}] + Cov[Z_{t-2}, Z_{t+p-1}] + \ldots \\
&+ Cov[Z_t, Z_{t+p-2}] + Cov[Z_{t-1}, Z_{t+p-2}] + Cov[Z_{t-2}, Z_{t+p-2}] + \ldots \\
&+ Cov[Z_t, Z_{t+1}] + Cov[Z_{t-1}, Z_{t+1}] + Cov[Z_{t-2}, Z_{t+1}] + \ldots
\end{aligned}
$$

Denote by $Lag_p = Cov[Z_t, Z_{t+p}]$. Expand the above equation and apply this new notation to get:

$$
\begin{aligned}
Cov[Y_t, Y_{t+p}] &= VarY_t \\
&\quad \begin{array}{cccccc}
+Lag_p & +Lag_{p+1} & +Lag_{p+2} & +Lag_{p+3} & +Lag_{p+4} & +\ldots \\
+Lag_{p-1} & +Lag_p & +Lag_{p+1} & +Lag_{p+2} & +Lag_{p+3} & +\ldots \\
+Lag_{p-2} & +Lag_{p-1} & +Lag_p & +Lag_{p+1} & +Lag_{p+2} & +\ldots \\
+Lag_{p-3} & +Lag_{p-2} & +Lag_{p-1} & +Lag_p & +Lag_{p+1} & +\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
+Lag_2 & +Lag_3 & +Lag_4 & +Lag_5 & +Lag_6 & +\ldots \\
+Lag_1 & +Lag_2 & +Lag_3 & +Lag_4 & +Lag_5 & +\ldots
\end{array} \\
&= VarY_t + \sum_{i=1}^{\infty} \min[i, p] Lag_i
\end{aligned}
$$

.

As we let $p \to \infty$, we get

$$VarY_t + \sum_{i=1}^{\infty} iCov[Z_t, Z_{t+i}] = 0.$$

Since Z_t is stationary, $Cov[Z_t, Z_{t+i}] = Cov[Z_t, Z_{t-i}]$. □

Proof of Proposition 2:

($\Leftarrow$) Given

$$\begin{aligned} EZ_t &= 0 \\ -2\sum_{p=1}^{\infty} Cov[Z_t, Z_{t-p}] &= VarZ_t \text{ and} \\ \sum_{p=1}^{\infty} pCov[Z_t, Z_{t-p}] &= C < \infty, \end{aligned}$$

it must be shown that Y_t is stationary.

First, we show that $E[Y_t]$ is a constant over time. Indeed, for any p we have

$$E[Y_t - Y_{t-p}] = E[Y_t - Y_{t-1} + Y_{t-1} - \ldots + Y_{t-p+1} - Y_{t-p}] = \tag{1}$$

$$\sum_{i=1}^{p} E[Z_{t-i+1}] = 0, \tag{2}$$

which implies $E[Y_t] = E[Y_{t-p}]$ for any p.

The variance of Y_t is given by

$$\begin{array}{llllll} \multicolumn{6}{l}{VarY_t = \text{Cov}[\sum_{l=-\infty}^{t} Z_l, \sum_{m=-\infty}^{t} Z_m] =} \\ +VarZ_t & +Lag_1 & +Lag_2 & +Lag_3 & +Lag_4 & +\ldots \\ +Lag_1 & +VarZ_t & +Lag_1 & +Lag_2 & +Lag_3 & +\ldots \\ +Lag_2 & +Lag_1 & +VarZ_t & +Lag_1 & +Lag_2 & +\ldots \\ +Lag_3 & +Lag_2 & +Lag_1 & +VarZ_t & +Lag_1 & +\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

We will add to each line $-\sum_{p=1}^{\infty} Cov[Z_t, Z_{t-p}]$ (we can do this since C is finite.) Now use $VarZ_t = -2\sum_{p=1}^{\infty} \text{Cov}[Z_t, Z_{t-p}]$ to get the result $VarY_t = \sum_{p=1}^{\infty} p\text{Cov}[Z_t, Z_{t-p}]$. Since $\sum_{p=1}^{\infty} pCov[Z_t, Z_{t-p}]$ is constant, it follows that the variance of Y_t is constant.

Now from the proof of proposition 1, we have that

$$Cov[Y_t, Y_{t-p}] = VarY_t + \sum_{i=1}^{\infty} min[i, p]Cov[Z_t, Z_{t-i}].$$

Obviously, $Cov[Y_t, Y_{t-p}]$ depends on p only since $VarY_t$ is constant over time. Hence, the process Y_t is stationary.

(⇒) Given that Y_t is stationary it must be shown that

$$\begin{aligned} -2\sum_{p=1}^{\infty} \text{Cov}[Z_t, Z_{t-p}] &= VarZ_t \text{ and} \\ \sum_{p=1}^{\infty} pCov[Z_t, Z_{t-p}] &= C < \infty. \end{aligned}$$

By assumption, $lim_{p\to\infty} Cov[Y_t, Y_{t-p}] = 0$. Use Proposition 1 to get $\sum_{p=1}^{\infty} pCov[Z_t, Z_{t-p}] = VarY_t$, which is a constant.

What is left to show is that $VarZ_t = -2\sum_{p=1}^{\infty} Cov[Z_t, Z_{t-p}]$. Fix k, and consider

$$\text{Cov}\left[\sum_{l=t-k}^{t} Z_l, \sum_{m=-\infty}^{t} Z_m\right].$$

Add $\sum_{p=1}^{k} Cov[Z_t, Z_{t-p}]$ to each term in $\sum_{l=t-k}^{t} Z_l$ to get

$$\begin{aligned} \text{Cov}\left[\sum_{l=t-k}^{t} Z_l, \sum_{m=-\infty}^{t} Z_m\right] &= -\sum_{p=1}^{k} p\text{Cov}[Z_t, Z_{t-p}] \\ &+ \sum_{m=1}^{k+1}\left[VarZ_t + \sum_{l=1}^{k} Cov[Z_t, Z_{t-l}] + \sum_{l=1}^{\infty} Cov[Z_t, Z_{t-l}]\right]. \end{aligned}$$

As $k \to \infty$ we have that

$$\begin{aligned} \lim_{k\to\infty} \text{Cov}\left[\sum_{l=t-k}^{t} Z_l, \sum_{m=-\infty}^{t} Z_m\right] &= VarY_t \text{ and} \\ \lim_{k\to\infty} -\sum_{p=1}^{k} p\text{Cov}[Z_t, Z_t - p] &= VarY_t \end{aligned}$$

hence,

$$\lim_{k\to\infty}(k+1)\left[VarZ_t + \sum_{l=1}^{k} \text{Cov}[Z_t, Z_{t-l}] + \sum_{l=1}^{\infty} Cov[Z_t, Z_{t-l}]\right] = 0,$$

which implies that $VarZ_t = -2\sum_{p=1}^{\infty} Cov[Z_t, Z_{t-p}]$. □

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Notes

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[2]The authors would like to thank Michael Cooper for helpful suggestions and comments.

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[4]The best-known index of Euronext Amsterdam, the AEX index, is made up of the 25 most active securities in the Netherlands. This index provides a fair representation of the Dutch economy.

[5]DAX 30 (Deutsche Aktien Xchange 30) is a Blue Chip stock market index consisting of the 30 major German companies trading on the Frankfurt Stock Exchange.

[6]The CAC 40, which takes its name from Paris Bourse's early automation system Cotation Assiste en Continu (Continuous Assisted Quotation), is a French stock market index. The index represents a capitalization-weighted measure of the 40 most significant values among the 100 highest market caps on the Paris Bourse.

[7]The FTSE 100 Index is a share index of the 100 most highly capitalized companies listed on the London Stock Exchange.

[8]Run down is the number of consecutive days with negative returns.

[9]Results with transaction costs are available from the authors upon request.

[10]Iraq war started on 3/19/2003.

Tables

Table 1: In-sample test results without transaction costs.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	6.09%	6.09%	6.09%	6.09%	6.09%
Worst Day	-4.64%	-4.06%	-4.06%	-4.06%	-4.06%
Percentage of Up Days	51.14%	52.50%	52.35%	53.41%	53.03%
Percentage of Down Days	48.86%	47.50%	47.65%	46.59%	46.97%
Average Daily Gain	0.57%	0.59%	0.60%	0.60%	0.59%
Standard Dev. of Positive Returns	9.97%	10.32%	10.58%	10.39%	10.20%
Average Daily Loss	-0.58%	-0.56%	-0.55%	-0.55%	-0.56%
Standard Dev. of Negative Returns	9.81%	9.40%	9.06%	9.28%	9.53%
Annual Return	2.83%	10.10%	12.90%	15.43%	11.70%
Standard Dev.	13.47%	13.46%	13.45%	13.44%	13.45%
Sharpe Ratio	0.21	0.75	0.96	1.15	0.87
Sortino Ratio	0.29	1.07	1.42	1.66	1.23
Skewness	0.12	0.36	0.48	0.39	0.32
Kurtosis	5.30	5.26	5.23	5.25	5.27
Average Run Down (days)	2	2	2	2	2
Standard Dev of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	8	9	8	8
Total Return	14.81%	52.91%	67.60%	80.82%	61.30%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the in-sample backtest without transaction costs as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 2: In-sample test results with transaction costs.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	6.09%	6.09%	6.09%	6.09%	6.09%
Worst Day	-4.64%	-4.06%	-4.06%	-4.06%	-4.06%
Percentage of Up Days	50.91%	52.27%	52.05%	53.18%	52.88%
Percentage of Down Days	49.09%	47.73%	47.95%	46.82%	47.12%
Average Daily Gain	0.57%	0.59%	0.60%	0.60%	0.59%
Standard Dev. of Positive Returns	9.97%	10.32%	10.59%	10.39%	10.21%
Average Daily Loss	-0.58%	-0.56%	-0.55%	-0.55%	-0.56%
Standard Dev. of Negative Returns	9.81%	9.40%	9.06%	9.28%	9.53%
Annual Return	2.33%	9.61%	12.41%	14.94%	11.21%
Standard Dev.	13.47%	13.46%	13.45%	13.44%	13.45%
Sharpe Ratio	0.17	0.71	0.92	1.11	0.83
Sortino Ratio	0.24	1.02	1.37	1.61	1.18
Skewness	0.12	0.36	0.48	0.39	0.32
Kurtosis	5.30	5.26	5.23	5.25	5.27
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	8	9	8	8
Total Return	12.22%	50.32%	65.00%	78.23%	58.71%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the in-sample backtest with transaction costs as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 3: Correlations between the strategy in-sample returns and the individual indexes.

	AEX	**CAC**	**DAX**	**FTSE**	**Strategy**
AEX	1.00				
CAC	0.94	1.00			
DAX	0.84	0.87	1.00		
FTSE	0.85	0.86	0.75	1.00	
Strategy	-0.06	-0.04	-0.01	-0.04	1.00

This table presents the correlations of the daily returns of the four indexes (AEX, CAC, DAX and FTSE), and the daily returns of the trading strategy (in-sample backtest) without transaction costs.

Table 4: Out-of-sample test without transaction cost, sliding window size 1000 days.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	3.77%	3.77%	3.84%	3.84%	3.84%
Worst Day	-4.05%	-4.05%	-4.05%	-4.05%	-4.05%
Percentage of Up Days	50.91%	52.73%	53.26%	52.42%	50.91%
Percentage of Down Days	49.09%	47.27%	46.74%	47.58%	49.09%
Average Daily Gain	0.48%	0.50%	0.52%	0.51%	0.49%
Standard Dev. of Positive Returns	7.46%	7.59%	7.99%	7.61%	7.51%
Average Daily Loss	-0.49%	-0.47%	-0.45%	-0.46%	-0.49%
Standard Dev. of Negative Returns	7.87%	7.75%	7.23%	7.70%	7.82%
Annual Return	2.00%	9.67%	16.31%	11.62%	2.21%
Standard Dev.	10.87%	10.86%	10.82%	10.85%	10.87%
Sharpe Ratio	0.18	0.89	1.51	1.07	0.20
Sortino Ratio	0.25	1.25	2.26	1.51	0.28
Skewness	-0.14	-0.12	0.20	-0.13	-0.15
Kurtosis	4.34	4.38	4.33	4.40	4.34
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	9	7	7	8	10
Total Return	10.50%	50.67%	85.45%	60.86%	11.57%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the out-of-sample backtest with sliding window size of 1000, without transaction costs, as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 5: Out-of-sample test without transaction costs, sliding window size 1250 days.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	3.78%	3.78%	3.86%	3.86%	3.86%
Worst Day	-3.99%	-3.99%	-3.99%	-3.99%	-3.99%
Percentage of Up Days	50.23%	52.65%	52.73%	52.65%	52.50%
Percentage of Down Days	49.77%	47.35%	47.27%	47.35%	47.50%
Average Daily Gain	0.48%	0.49%	0.50%	0.49%	0.47%
Standard Dev. of Positive Returns	7.55%	7.50%	7.88%	7.50%	7.40%
Average Daily Loss	-0.46%	-0.46%	-0.44%	-0.45%	-0.47%
Standard Dev. of Negative Returns	7.64%	7.68%	7.22%	7.69%	7.80%
Annual Return	3.20%	10.07%	14.82%	10.38%	6.43%
Standard Dev.	10.66%	10.64%	10.62%	10.64%	10.65%
Sharpe Ratio	0.30	0.95	1.40	0.98	0.60
Sortino Ratio	0.42	1.31	2.05	1.35	0.83
Skewness	-0.06	-0.11	0.21	-0.11	-0.15
Kurtosis	4.74	4.79	4.73	4.80	4.77
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	7	10	7	10
Total Return	16.79%	52.76%	77.65%	54.38%	33.69%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the out-of-sample backtest with sliding window size of 1250 days, without transaction costs, as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 6: Out-of-sample test without transaction cost, sliding window size 1500 days.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	3.79%	3.79%	3.92%	3.92%	3.92%
Worst Day	-3.92%	-3.92%	-3.90%	-3.90%	-3.90%
Percentage of Up Days	50.91%	54.09%	53.94%	53.56%	52.65%
Percentage of Down Days	49.09%	45.91%	46.06%	46.44%	47.35%
Average Daily Gain	0.48%	0.48%	0.50%	0.48%	0.49%
Standard Dev. of Positive Returns	7.94%	7.93%	8.21%	7.69%	8.11%
Average Daily Loss	-0.46%	-0.45%	-0.44%	-0.46%	-0.45%
Standard Dev. of Negative Returns	7.69%	7.68%	7.31%	7.97%	7.47%
Annual Return	4.99%	13.54%	16.41%	10.01%	12.23%
Standard Dev.	10.81%	10.78%	10.76%	10.79%	10.79%
Sharpe Ratio	0.46	1.26	1.52	0.93	1.13
Sortino Ratio	0.65	1.76	2.24	1.26	1.64
Skewness	0.05	0.00	0.27	-0.13	0.21
Kurtosis	5.20	5.27	5.19	5.27	5.19
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	8	10	8	10
Total Return	26.14%	70.91%	85.93%	52.42%	64.04%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the out-of-sample backtest with sliding window size of 1500 days, without transaction costs, as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 7: Correlations between the strategy out-of-sample returns and the individual indexes.

	AEX	**CAC**	**DAX**	**FTSE**	**Strategy**
AEX	1.00				
CAC	0.94	1.00			
DAX	0.84	0.87	1.00		
FTSE	0.85	0.86	0.75	1.00	
Strategy	-0.02	-0.03	0.01	-0.04	1.00

This table presents the correlations of the daily returns of the four indexes (AEX, CAC, DAX and FTSE), and the daily returns of the trading strategy out-of-sample backtest, lag parameter of 25 and window size 1000 days, without transaction costs.

Table 8: Out-of-sample test without transaction cost, cumulative window, starting with window size 1000 days.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	3.80%	3.80%	3.88%	3.88%	3.88%
Worst Day	-4.03%	-4.03%	-4.03%	-4.03%	-4.03%
Percentage of Up Days	50.30%	52.12%	52.65%	53.11%	51.52%
Percentage of Down Days	49.70%	47.88%	47.35%	46.89%	48.48%
Average Daily Gain	0.47%	0.47%	0.49%	0.47%	0.46%
Standard Dev. of Positive Returns	7.41%	7.44%	7.89%	7.42%	7.46%
Average Daily Loss	-0.45%	-0.45%	-0.43%	-0.45%	-0.46%
Standard Dev. of Negative Returns	7.71%	7.69%	7.15%	7.72%	7.68%
Annual Return	3.36%	8.51%	14.13%	10.49%	4.56%
Standard Dev.	10.52%	10.51%	10.49%	10.50%	10.52%
Sharpe Ratio	0.32	0.81	1.35	1.00	0.43
Sortino Ratio	0.44	1.11	1.98	1.36	0.59
Skewness	-0.14	-0.15	0.22	-0.17	-0.10
Kurtosis	5.13	5.17	5.11	5.20	5.13
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	9	8	8	8
Total Return	17.59%	44.57%	74.01%	54.95%	23.89%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the out-of-sample backtest, cumulative window, starting with window size 1000 days, without transaction costs, as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 9: Out-of-sample test without transaction cost, cumulative window, starting with window size 1250 days.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	3.76%	3.76%	3.88%	3.88%	3.88%
Worst Day	-3.93%	-3.93%	-3.93%	-3.93%	-3.93%
Percentage of Up Days	50.23%	53.64%	53.26%	52.88%	52.27%
Percentage of Down Days	49.77%	46.36%	46.74%	47.12%	47.73%
Average Daily Gain	0.49%	0.49%	0.50%	0.48%	0.48%
Standard Dev. of Positive Returns	7.74%	7.67%	7.96%	7.46%	7.61%
Average Daily Loss	-0.46%	-0.46%	-0.45%	-0.47%	-0.47%
Standard Dev. of Negative Returns	7.55%	7.61%	7.25%	7.85%	7.68%
Annual Return	3.29%	12.97%	13.65%	8.33%	7.05%
Standard Dev.	10.74%	10.71%	10.70%	10.73%	10.73%
Sharpe Ratio	0.31	1.21	1.28	0.78	0.66
Sortino Ratio	0.44	1.70	1.88	1.06	0.92
Skewness	0.05	-0.07	0.22	-0.17	-0.06
Kurtosis	4.91	4.99	4.90	4.96	4.94
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	7	7	7	7
Total Return	17.22%	67.92%	71.51%	43.65%	36.94%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the out-of-sample backtest, cumulative window, starting with window size of 1250 days, without transaction costs, as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

Table 10: Out-of-sample test without transaction cost, cumulative window, staring with window size 1500 days.

Performance Measures	Lag Parameter P				
	10	**20**	**25**	**30**	**40**
Best Day	3.91%	4.07%	4.07%	4.07%	4.07%
Worst Day	-4.37%	-4.37%	-4.37%	-4.37%	-4.37%
Percentage of Up Days	49.92%	52.80%	52.88%	53.18%	51.97%
Percentage of Down Days	50.08%	47.20%	47.12%	46.82%	48.03%
Average Daily Gain	0.50%	0.51%	0.52%	0.50%	0.52%
Standard Dev. of Positive Returns	7.92%	8.32%	8.42%	8.19%	8.68%
Average Daily Loss	-0.49%	-0.47%	-0.46%	-0.47%	-0.46%
Standard Dev. of Negative Returns	8.30%	7.85%	7.71%	8.01%	7.41%
Annual Return	1.09%	12.23%	14.43%	11.42%	12.71%
Standard Dev.	11.24%	11.21%	11.20%	11.21%	11.21%
Sharpe Ratio	0.10	1.09	1.29	1.02	1.13
Sortino Ratio	0.13	1.56	1.87	1.43	1.72
Skewness	-0.17	0.13	0.17	0.03	0.39
Kurtosis	5.20	5.21	5.21	5.23	5.13
Average Run Down (days)	2	2	2	2	2
Standard Dev. of Run Down (days)	1	1	1	1	1
Max Run Down (days)	8	8	8	8	8
Total Return	5.70%	64.05%	75.61%	59.82%	66.57%
Days Traded	1320	1320	1320	1320	1320

This table presents eighteen performance measures for the out-of-sample backtest, cumulative window, starting with window size of 1500 days, without transaction costs, as functions of the lag parameter P. The *Best Day* is the highest daily return observed, and the *Worst Day* is the lowest. The *Percentage of Up Days* is the fraction of days with positive returns, whereas the *Percentage of Down Days* corresponds to the negative returns. The *Average Daily Gain* is the average of the positive daily return, and the *Standard Deviation of Positive Returns* is the annualized volatility of the positive daily returns. The *Average Daily Loss* is the average of the negative daily returns with the corresponding *Standard Deviation of Negative Returns* (annualized). The *Annual Average* is the annualized average daily return. The *Standard Deviation* is the annualized volatility of the daily returns. The *Sharpe* ratio equals to the annualized average return divided by annualized volatility. The *Sortino* ratio equals to the annualized average return divided by annualized standard deviation of the negative returns. The *Skewness* and *Kurtosis* are standard statistical measures for the degree of asymmetry of the distribution of the daily returns. The Run Down corresponds to the number of consecutive days with negative returns, and the *Average Run Down*, *Standard Deviation of Run Down* and *Maximum Run Down* are the corresponding average, volatility, and maximum. The *Total Return* is the cumulative total return of the strategy, and *Days Traded* are the total number of trading days in this backtest. The values of the lag parameter are window sizes of 10, 20, 25, 30, and 40 days.

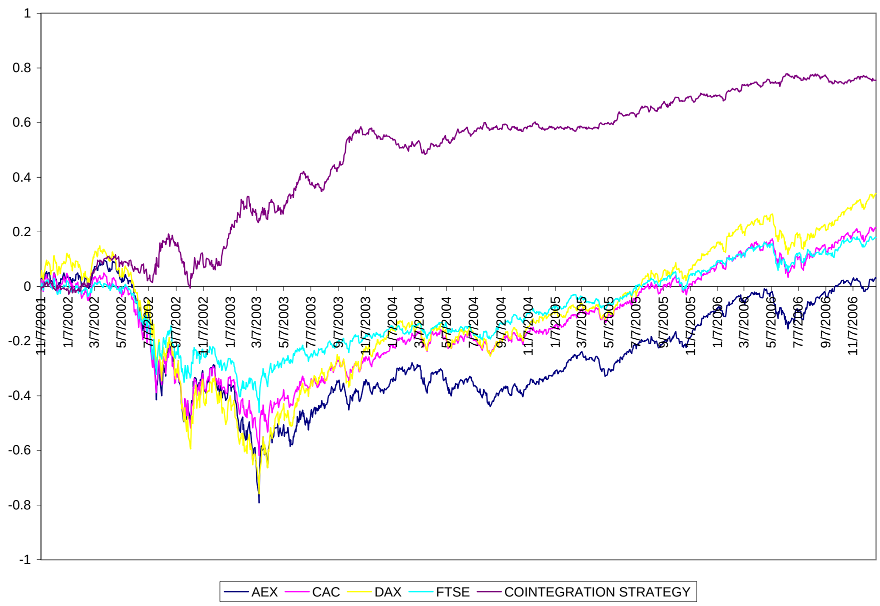
Table 11: Correlations of the daily returns of the strategy with the daily returns of the four indexes when the lag parameter is 25 days and the cumulative window starts at 1500 days.

	AEX	**CAC**	**DAX**	**FTSE**	**Strategy**
AEX	1.00				
CAC	0.94	1.00			
DAX	0.84	0.87	1.00		
FTSE	0.85	0.86	0.75	1.00	
Strategy	-0.04	-0.04	0.02	-0.03	1.00

This table presents the correlations of the daily returns of the four indexes (AEX, CAC, DAX and FTSE), and the daily returns of the trading strategy out-of-sample backtest, lag parameter of 25 and cumulative window size starting at 1500 days, without transaction costs.

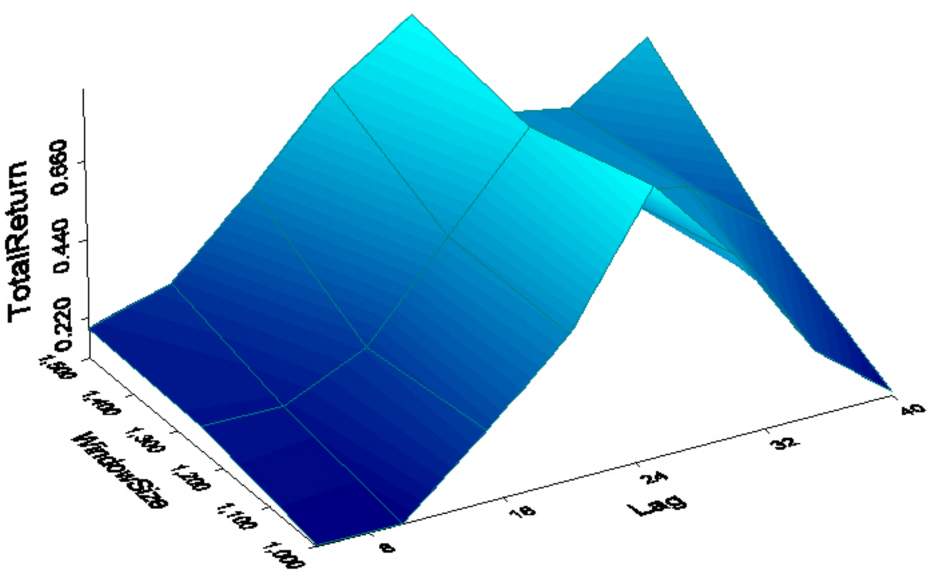
Figures

Figure 1: Total return for the in-sample backtest vs the four indexes.



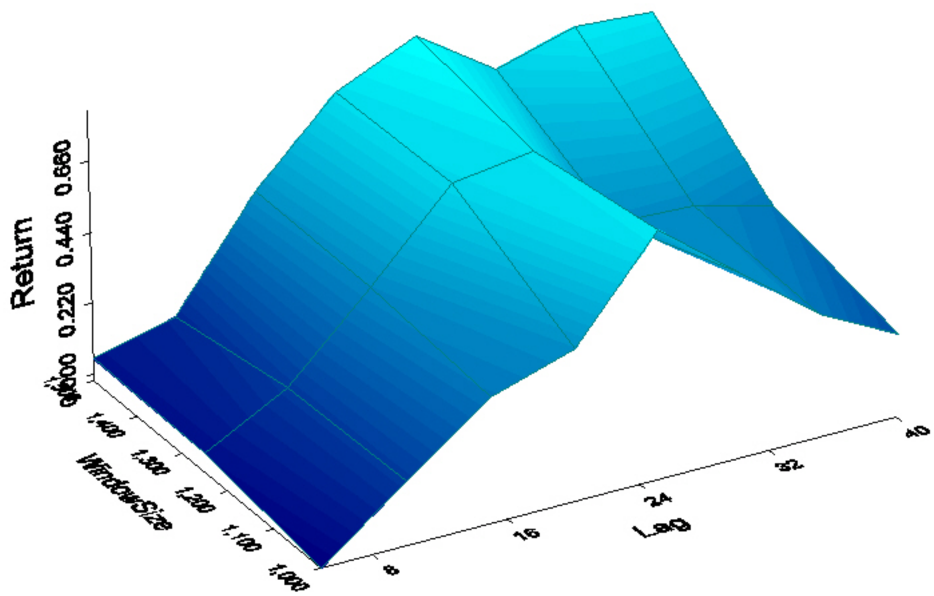
This figure shows the dynamics of the cumulative returns from the in-sample backtest without transaction costs, compared to the corresponding returns of the four indexes (AEX, CAC, DAX, FTSE). The horizontal axis is the time, and the vertical axis is the total cumulative return.

Figure 2: Total return for out-of-sample test as a function of the lag parameter and window size, fixed case, no transaction costs.



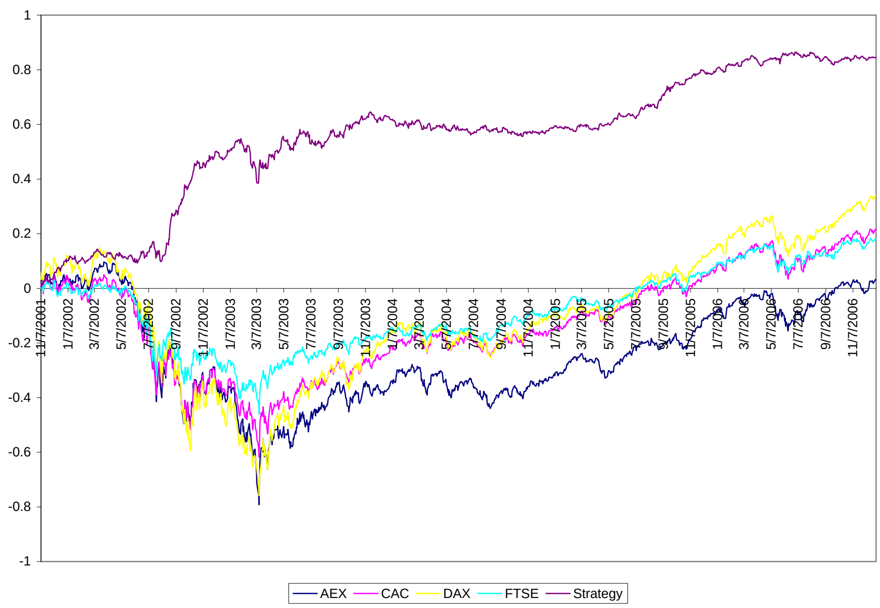
This figure is a 3 dimensional plot of the total return of the cointegration daily strategy (out-of-sample test) as a function of the window size and the lag. The strategy was run using the following values of the window size 1000, 1250 and 1500; the values of the lag were 10,15,20,25,30,35 and 40.

Figure 3: Total return for out-of-sample aggregate test as a function of the lag parameter and window size, no transaction costs.



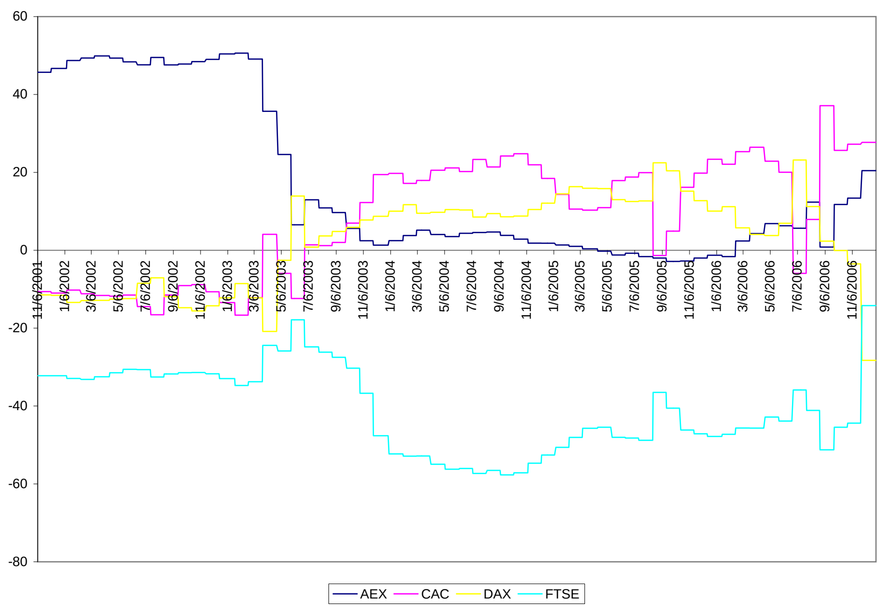
This figure is a 3 dimensional plot of the total return of the cointegration daily strategy (out-of-sample aggregate test) as a function of the window size and the lag. The strategy was run using the following values of the window size 1000, 1250 and 1500; the values of the lag were 10,15,20,25,30,35 and 40.

Figure 4: Total return for the out-of-sample backtest (lag 25, window size 1000) vs the four indexes.



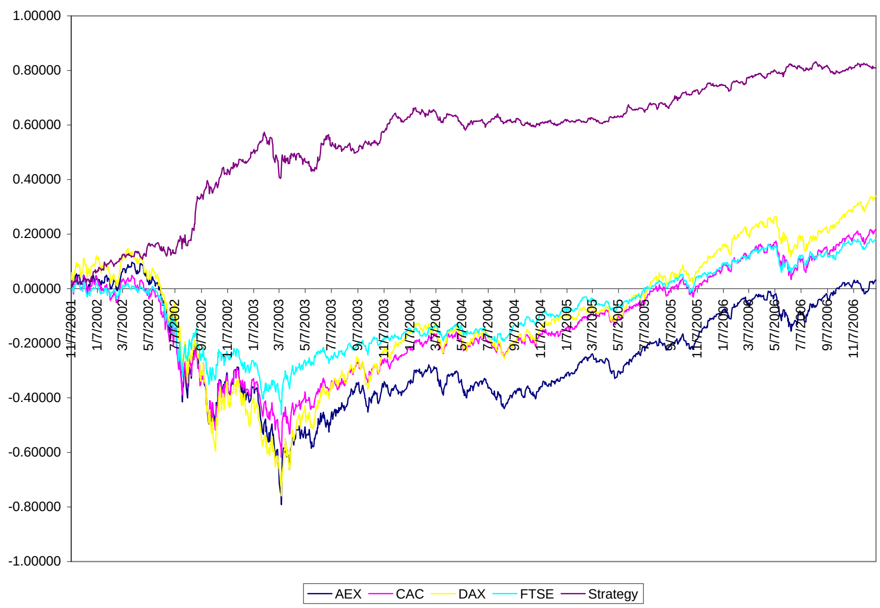
This figure shows the dynamics of the cumulative returns from the out-of-sample backtest done with a lag of 25 days and a window size of 1000 days without transaction costs, compared to the corresponding returns of the four indexes (AEX, CAC, DAX, FTSE). The horizontal axis is the time, and the vertical axis is the total cumulative return.

Figure 5: Time plot of the estimated cointegration vector.



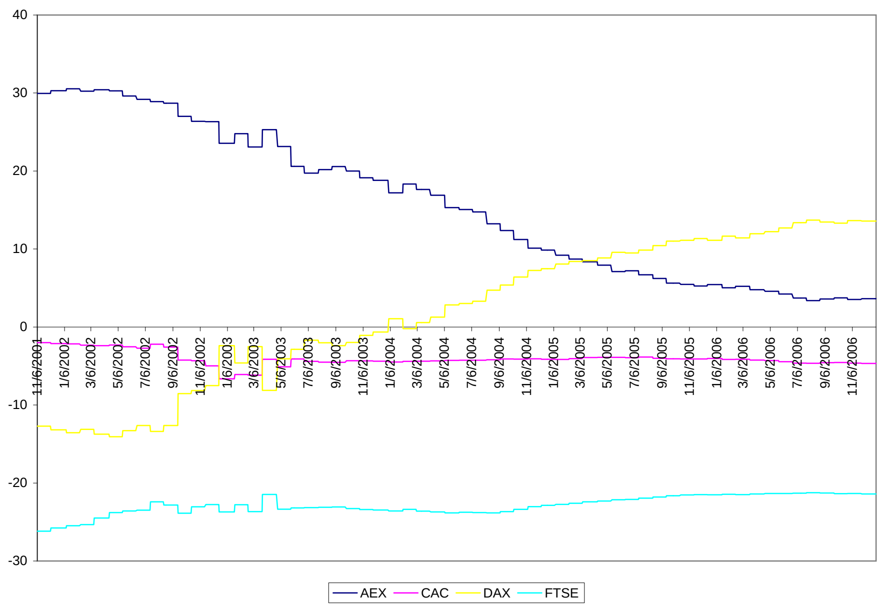
This figure shows the estimated cointegration vector as a function of time. The vector is re-estimated every 22 days using the Johansen cointegration rank test with data from the previous 1000 days.

Figure 6: Total return for the out-of-sample backtest (lag 25, cumulative window starting at 1500) vs the four indexes.



This figure shows the dynamics of the cumulative returns from the out-of-sample backtest done with a lag of 25 days and a cumulative window starting at 1500 days without transaction costs, compared to the corresponding returns of the four indexes (AEX, CAC, DAX, FTSE.) The horizontal axis is the time, and the vertical axis is the total cumulative return.

Figure 7: Time plot of the estimated cointegration vector when the lag parameter is 25 days, and the cumulative window starts at 1500 days.



This figure shows the estimated cointegration vector as a function of time. The vector is re-estimated every 22 days using the Johansen cointegration rank test using all data up to the current time period