Exercise 14
Interest Rates in Binomial Grids

Financial Models in Excel, F65/F65D

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The objective with this exercise is to introduce the methodology needed to price callable bonds. The methodology will be useful for pricing of interest rate dependent derivatives in general, however. To price a callable bond, we need to price an American type option. As for stock options, we use binomial grids. Although the setup looks like the stock option setup at first sight, binomial grids for interest rate assets are much more complicated than for stocks, and we will only scratch the surface. If you want to proceed, you might consult *Fixed Income Securities* by Bruce Tuckman, Wiley or *Fixed-income securities* by Martellini, Priaulet, and Priaulet, 2003, Wiley.

1. INTRODUCTION

The issuer of a callable bond have the right to buy back the bond at a given price, usually whenever he wants.[1]. I.e., the buyer of the bond receives a bond, but sells an American put option at the same time. As in the stock option case, there will not be closed form solutions for American puts. Hence, we must set up binomial grids for how the interest rate might evolve in the future in order to determine the probability that the bond will be called in the future.

Interest rate modeling is much more complicated than modeling stock prices, although it does not seem that way at first sight. I guess you can point to two main reasons. First of all, our interest rate grid should match the observed term structure. To do this without any inconsistencies turn out to be rather complicated. Second, (and somewhat related) interest rates are not random walks like stock prices. If we observe a relatively high short interest rate today, it is perfectly legitimate to assume that the interest rate will fall in the future. It all depends on your "model" for how the interest rate evolves. And there are a lot of competing models. Some models are special cases of the following specification:

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[*]Inspiration from Ken Bechmann's lecture-notes is acknowledged.

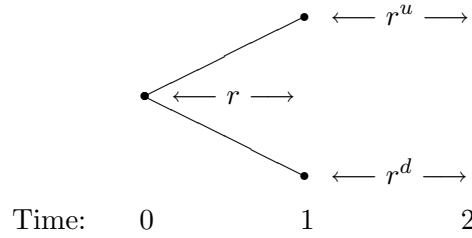[1]In practice, you can only do it a few dates a year in Denmark, (Bermudan)

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Figure 1: The evolution of one-period interest rates in a two period grid.

$$r_t = r_{t-\tau}e^{m_t\tau+\sigma_t\sqrt{\tau}\epsilon_t}, \quad \epsilon_t \sim IID(0,1),$$

where τ is the period length which we will set to one to simplify:

$$r_t = r_{t-1}e^{m_t+\sigma_t\epsilon_t}.$$

In general, m_t and σ_t can be be complicated functions and different specifications have been suggested by different authors. Below, we will start out with something close to the "Black-Derman-Toy"-model where ϵ takes only two values, -1 and 1 with equal probability:

$$(1) \qquad r_t = \begin{cases} r_{t-1}e^{m_t+\sigma_t}, & p = \frac{1}{2} \\ r_{t-1}e^{m_t-\sigma_t}, & (1-p) = \frac{1}{2} \end{cases}$$

A central and critical assumption below is that p and $(1-p)$ are "objective" probabilities and that the investor is risk neutral. If you do not like this assumption, you might interpret the probabilities as risk-adjusted probabilities similar to q^u and q^d we derived for the stock grids. In interest rate grids, however, it is much harder to derive q^u and q^d from observed prices,[2] so I will stick to the risk-neutral investor interpretation.

2. Two period-grids

In the following, let r denote the one period interest rate observed at the date of the analysis $(t = 0)$ and let r^u and r^d denote the two possible values of the one period interest rate observed at time $t = 1$, see Figure 1.

According to the interest rate model we have assumed, this gives us the following values

$$(2) \qquad \begin{aligned} r^u &= re^{m_1+\sigma_1} \\ r^d &= re^{m_1-\sigma_1}. \end{aligned}$$

where the one period rate for the first period r is the observed one-period zero-coupon interest rate n_1.

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[2]This is actually a third reason why interest rate grids are more complicated than stock grids.

$$FV^u = \frac{105}{1+r^u}$$

$$\frac{\frac{1}{2}(FV^u+FV^d)+5}{1+r}$$

$$FV^d = \frac{105}{1+r^d}$$

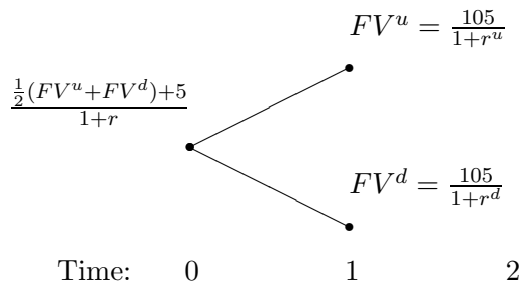Time: 0   1   2

Figure 2: The face value of the normal bond.

Exercise 14.1.a: Assume that the period length of our grids is one year and assume that the zero-coupon term structure is known, (see my solution for data on the US term structure).

- Determine r^u and r^d when $m_1 = 0$, $\sigma_1 = 0.3 \sim 30\%$, and $r = n_1$.

We are now ready to price bonds with our two-period grid. Assume that we want to price a two-period bond paying a coupon equal to 5 each year (at time $t = 1$ and $t = 2$) and a principal equal to 100 which is payed at time $t = 2$.

Since the payoff is not state dependent, there is no reason to use a binomial grid to price this bond, but let's do it anyway. From figure 2 you should be able to see what goes on.[3] In the r^u-state, the face value of the bond is equal to the cash flow received at maturity (equals 105) discounted with r^u. The FV^d is determined in a similar way. The price of the bond at $t = 0$ is the expected face value (remember: risk neutral agents) at time $t = 1$ plus the coupon discounted with the relevant interest rate r.

Exercise 14.1.b: Consider a bond similar to the one described above but with a coupon equal to 1.5 instead. I.e., Cash flow equal to 1.5 at $t = 1$ and 101.5 at $t = 2$.

- Price the bond using the interest rate grid determined in Exercise 14.1.a.

When we know how to price normal bonds in the grid, it is easy to proceed to callable bonds. Assume that the bond described above can be called at a price of 100. This implies that if the face value of the bond exceeds 100, the issuer will exercise his American put, and force the owner of the bond to sell it back to the issuer for a price of 100. As a result, the face value can never exceed 100. Figure (3) shows how to take this into account when the present value of the bond is determined.

Exercise 14.1.c: Consider the bond described in Exercise 14.1.b but now, the bond can be called by the issuer at a price of 100.

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[3]I would prefer to use continuously compounding as we used in our stock-grids, since it makes it easy to split the grid into smaller periods, but most other text books uses discrete time compounding (including the two references above) so I will just play along...
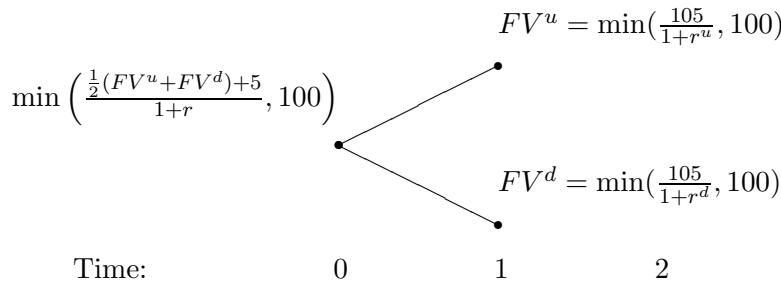
$$FV^u = \min\left(\tfrac{105}{1+r^u}, 100\right)$$

$$\min\left(\frac{\tfrac{1}{2}(FV^u + FV^d) + 5}{1+r}, 100\right)$$

$$FV^d = \min\left(\tfrac{105}{1+r^d}, 100\right)$$

Time: 0   1   2

Figure 3: The face value of a callable bond.

- Price the bond using the interest rate grid determined in Exercise 14.1.a. and compare with the results in Exercise 14.1.c.
- Set up a grid that explicitly determines the exercise policy for the issuer.

2.1. CALIBRATION W.R.T. THE TERM STRUCTURE

It is probably not right to assume that $m_1 = 0$ as we did in the previous section. One way around the problem would be to estimate a model for how the expected value of the short rate changes over time. Another approach, which is followed here, is to calibrate the value of m_1 such that the interest rates in our grid fits the observed term structure in some sense.

Note first that if we combine the two expressions in (2), we get the following relation:

$$(3) \qquad r^u = e^{2\sigma_1} r^d.$$

I.e., if we can just determine r^d we also have the "up"-interest rate. If we know the drift rate m_1, we can determine r^d with (2), but instead of determining m_1 directly, we will use the observed term structure to determine the right level of r^d, (when we have r^d, we also have r^u by (3)).

We will determine r^d by impose that the price of a two period zero coupon bond implied by our grid should be the same as the one implied by the observed two-period zero coupon interest rate, n_2. Since our investor is risk-neutral by assumption, we get

$$(4) \qquad \frac{1}{(1+n_2)^2} = \frac{1}{(1+r)}\left(\frac{1}{2}\frac{1}{1+r^u} + \frac{1}{2}\frac{1}{1+r^d}\right).$$

The interpretation is straight forward, (I hope!). Another way to get some intuition is to recognize that discounting with n_2 in two periods is the same as discounting with n_1 in the first period and with the forward rate $f_{1,2}$ between $t = 1$ and $t = 2$:

$$\frac{1}{(1+n_1)(1+f_{1,2})} = \frac{1}{(1+n_1)}\left(\frac{1}{2}\frac{1}{1+r^u} + \frac{1}{2}\frac{1}{1+r^d}\right)$$

(5)
$$\Downarrow$$

$$\frac{1}{1+f_{1,2}} = \frac{1}{2}\frac{1}{1+r^u} + \frac{1}{2}\frac{1}{1+r^d}.$$

The interpretation is that the expected discount factor of our grid for the period $t = 1$ to $t = 2$ should be equal to the discount factor implied by the forward rate.[4]

Note that if we use (3) to get rid of r^u there is only one unknown in (5): r^d. You can find solutions with pen and paper or use the Solver.

Before we move to the first exercise, I would like to introduce notation for the one-period discount factors:

$$d = \frac{1}{1+r}, \quad d^u = \frac{1}{1+r^u}, \quad d^d = \frac{1}{1+r^d}, \quad \text{etc.}$$

With this notation, we can, for instance, get a simpler look for the restriction in (4),

$$\frac{1}{(1+n_2)^2} = \frac{1}{2}dd^u + \frac{1}{2}dd^d,$$

which is also easier to work with in Excel.

> **Exercise 14.1.d:**   In my solution you will find a relevant value for n_2 for the US interest rate.
>
> - Determine the interest rates as in Exercise 14.1.a, but this time the choice of r^u and r^d should obey the restrictions in (3) and (4).
>
> - Determine the new prices of the bonds from Exercise 14.1.b and 14.1.c. Compare with previous results.

3.   THREE PERIODS AND MULTI-PERIOD GRIDS

When we constructed the binomial grids for stocks, we used the same method for constructing the subsequent stock moves as the method used to construct the moves for the first period. Things are not that simple when it comes to interest rate grids. Therefore, we will proceed slowly, and consider the three period case in details. Once we master the technique and ideas behind the three period grid, where will be no conceptual complexity added when we move to multiple period grids in general. But the computational complexity will increase, of course. To fix ideas, Figure 4 show the one-period rates we have to determine. The first three, r, r^u, and r^d have already been determined. We are now left with the problem of determining three (four) new rates.

We will still be using the setup in (1) but whereas we had only one value for the drift rate m_1 and the volatility σ_1 at time $t = 1$, the drift rate and the volatility

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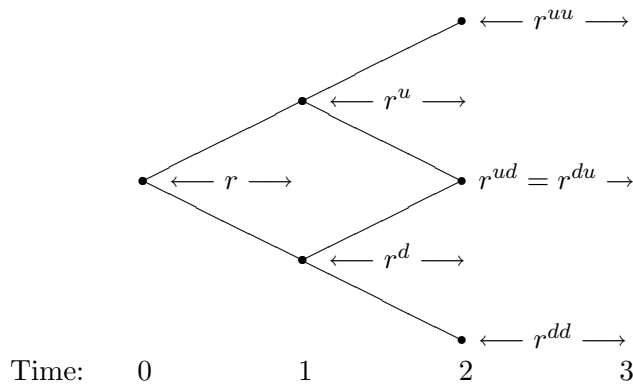[4]Warning: This intuition will not apply in general to grids with more than two periods

Figure 4: The evolution of one-period interest rates in a three period grid.

for the interests of the last period, m_2 and σ_2, might depend on which state we are in at time $t = 1$. For instance, if we observe a high interest rate r^u we might expect that future interest rates will have a stronger tendency to decrease in the future than if we observed a low interest rate, r^d. If this is the case, we might have something like $m_2^u < m_2^d$. Such features might have an unpleasant implication, that we might not be sure that our interest tree is recombining. To see this, use (1) to get the expressions for the two interest rates which we would like to combine:

$$r^{ud} = re^{m_1+\sigma_1+m_2^u-\sigma_2^u}$$
$$r^{du} = re^{m_1-\sigma_1+m_2^d+\sigma_2^d}.$$

If we want $r^{ud} = r^{du}$, certain relationships between the m and the σ-values have to hold. We will try to avoid this discussion for a moment by assuming that the volatility is the same for all states, σ, and that the m-values are chosen to make our trees recombine. As you will see now, we will not pay explicit attention to how the m-values are chosen.

3.1. CONSTANT VOLATILITY

We know that from (3) that $r^u = e^{2\sigma}r^d$. Using the same kind of arguments, we can show that

(6)
$$r^{du} = e^{2\sigma}r^{dd} \quad \text{and}$$
$$r^{uu} = e^{2\sigma}r^{ud}$$

and since we have assumed a recombining tree, $r^{ud} = r^{du}$, and a constant volatility we can also express r^{uu} in using only σ and r^{dd}:

$$r^{uu} = e^{4\sigma}r^{dd} \quad \text{and}$$

Result: We only need to determine r^{dd}. Then, the values for $r^{ud} = r^{du}$ and r^{uu} follows.

Following the same procedure as before, we will choose r^{dd} such that the price of a 3 period zero coupon bond is the same whether we use our grid or the observed zero coupon interest rates to price with. You can express this restriction explicitly as (I'm using discount factors instead of interest rates to simplify)

$$(7) \qquad \frac{1}{(1+n_3)^3} = \frac{1}{4}d \cdot d^u \cdot d^{uu} + \frac{1}{4}d \cdot d^u \cdot d^{ud} + \frac{1}{4}d \cdot d^d \cdot d^{du} + \frac{1}{4}d \cdot d^d \cdot d^{dd}$$

but it is probably easier to impose the restriction using recursive pricing in a grid in Excel.

> **Exercise 14.2.a:** We will try to expand the grid determined in Exercise 14.1.d with one period. Hence, we use the old values for r, r^u, and r^d. To find the new state-dependent interest rates you need n_3 which can be found in my solution. As before, we assume $\sigma = 0.3 = 30\%$, (in both states, according to our assumption above). Also we assume a recombining grid.
>
> - Determine r^{uu}, r^{ud}, and r^{dd} according to (6) and (7).

MORE PERIODS

Once you have tried to expand the grid from two to three periods, there is nothing new in expanding from three till four periods (check out Figure 5 to figure out my notation):

- Copy the three period grid of interest rates

- Assume recombining grid and constant volatility over states:

$$r^{3u} = e^{2\sigma}r^{2ud}$$
$$r^{2ud} = e^{2\sigma}r^{u2d}$$
$$r^{u2d} = e^{2\sigma}r^{3d}$$

  which leaves only one degree of freedom, (I usually consider the lowest one, here r^{3d}, as the free one).

- Choose r^{3d} such that the price of a zero-coupon bond priced by the grid is equal to the price implied by the relevant zero-coupon interest rate, here n_4.

The last step will have to be solved using the Solver. Once you understand the method, you should be able to proceed with as many periods as you want.

> **Exercise 14.2.b:** Copy the interest rate grid from Exercise 14.2.a and maintain $\sigma = 0.3$ for all periods. Use n_4 from my solution.
>
> - Expand the interest rate grid to a four period grid.
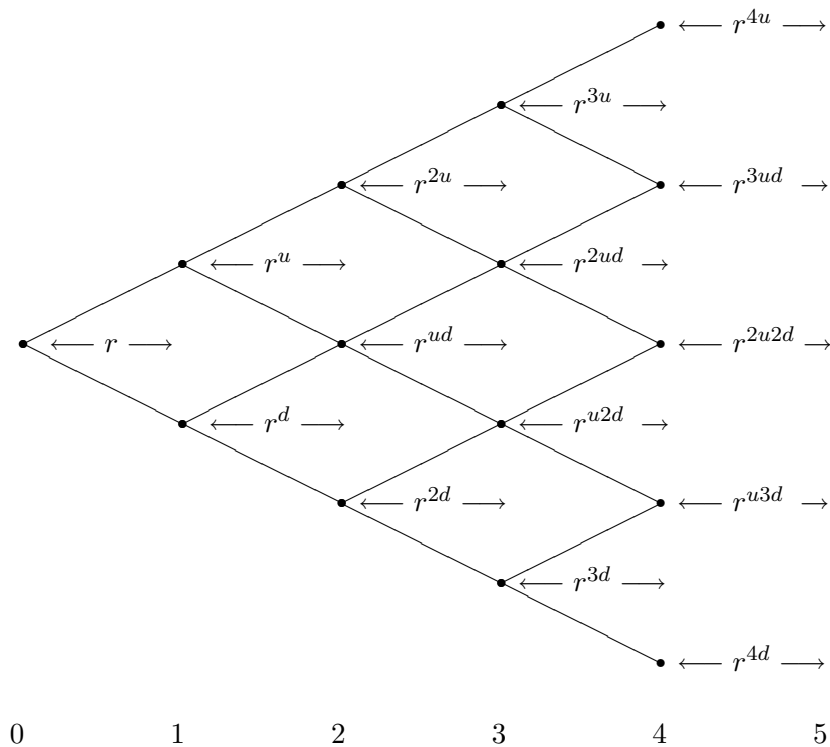> - Expand the interest rate grid to a five period grid.

Figure 5: The evolution of one-period interest rates in a five period grid.

Exercise 14.2.c: Consider a 5 year bond that pays a coupon at 3 at $t \in \{1, 2, \cdots 5\}$ and a principal at 100 at $t = 5$.

- Determine the price of the bond.
- Determine the price of the bond if it can be called at a price of 100.
- Determine the exercise policy for the callable bond.

After Exercise 14.2.b you have probably figured out that in order to construct a 20 period interest rate grid in Excel, you have to put in a lot of work and use a lot of cells. For each new period, you have to set up a new zero-coupon price grid and calibrate the level using the Solver.

It would be nice to have a VBA function for constructing these grids. Unfortunately, it is not straight forward to call the Solver from within VBA, as you might remember.[5]

However, the finding the right r^d, r^{dd}, r^{3d}, r^{4d}, etc. is a well-behaved problem and we can use the same method we used to program our own "Solver" when we determined implied volatilities. Check out the relevant Exercise or page 303 in Benninga if you forgot.

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[5]As noted before, it IS possible, but it is beyond the scope of the course.

Exercise 14.3 Although the VBA code is relatively simple, (no new concepts) the first part of the exercise is probably a bit complicated, (I guess?).

- Program a VBA-function which can determine the binomial interest rate grids of arbitrary size, given the assumptions made above.

  Note: Input should be a vector of relevant zero coupon interest rates and the volatility parameter σ.

- Price a bond with maturity in 20 years, a coupon each year equal to 3, and a principal at 100 payed at $t = 100$.

- Price the bond again, but this time it is assumed that the bond can be called at a price equal to 100.

- Determine the value of the implicit American put due to the callable-property of the bond.

We will end the mandatory part of the exercise here, but notice that you can use the grids you have constructed to price all sorts of interest rate dependent derivatives, like interest rate options, caps, floors etc.

If you are only interested in the stuff relevant for the exam, you should stop here. I made the rest of the exercise out of my own curiosity, and I'm not sure it is relevant!

<div align="center">

Thank you for attending the course
Good luck
Peter

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4.   Some Inconsistencies of the Grid, (Optional)

Although the interest rate grids constructed above seem very consistent at first sight, it is easy to show inconsistencies for grids with more than two periods. For instance, the interest rates for the period from $t = 2$ to $t = 3$ is not consistent with the forward rate $f_{2,3}$ for the same period.
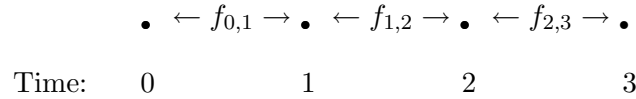
$$\bullet \quad \leftarrow f_{0,1} \rightarrow \bullet \quad \leftarrow f_{1,2} \rightarrow \bullet \quad \leftarrow f_{2,3} \rightarrow \bullet$$

Time: 0   1   2   3

Figure 6: The forward rate notation.

To show this, we need the probabilities the states at $t = 2$. Since we have maintain the assumption of equal probability for "up" and "down". As a result, we get the probabilities in Figure 7. Note that the likelihood of the middle state "ud" is twice as high as the others, since there are two ways of arriving in the middle state, but only one way of arriving in the others. And since all paths have the same probability, the result follows. In the three state case, it is still easy to determine how many paths end up in a given state, but when we consider the general case, we will have to use some formulas, which are available in Excel.

Now suppose we have a cash-flow at time $t = 3$ equal to 1, which we would like to transform into a time $t = 2$-cash flow. Then, the expected cash flow of the following two strategies should be the same: 1) Engage in a forward contract at time $t = 0$ which transform $t = 3$ cash flow into $t = 2$ cash flow with the forward rate $f_{2,3}$ or 2) wait until time $t = 2$, and then discount the future cash flow with the state-dependent interest rate. We can express the equilibrium condition as
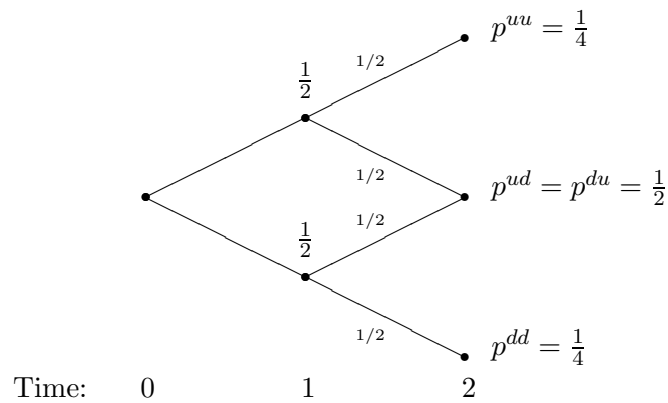

Figure 7: The probability of different states

$$
(8) \quad \frac{1}{1 + f_{2,3}} = \frac{p^{uu}}{1 + r^{uu}} + \frac{p^{ud}}{1 + r^{ud}} + \frac{p^{dd}}{1 + r^{dd}}
$$

$$
= \frac{1}{4(1 + r^{uu})} + \frac{1}{2(1 + r^{du})} + \frac{1}{4(1 + r^{dd})}.
$$

If the expected cash flows of the two strategies are not the same, we will not be in equilibrium if the investors are risk-neutral.

> **Exercise 14.4** Consider the three period interest rate grid determined in Exercise 14.2.a, and assume that you want to transform a cash flow at 100 from time $t = 3$ to $t = 2$.
>
>   - Check if the equilibrium condition in (8) holds. I.e., does both strategies provide the same expected cash flow at time $t = 2$?

AN EXPLANATION

In the Exercise above, you will only find small inconsistencies, but for longer horizons they will be more significant. The reason for the inconsistencies can best be seen by considering first how define the implicit forward rates,

$$
(9) \quad (1 + n_1)(1 + f_{1,2})(1 + f_{2,3}) = (1 + n_3)^3.
$$

When we constructed our grids, we imposed the same restriction on our r's, but since our interest rates are stochastic, we did it in expected values:

$$
(10) \quad E\left((1 + r)(1 + r_{1,2})(1 + r_{2,3})\right) = (1 + n_3)^3.
$$

where $r_{1,2}$ denote the stochastic interest rate in our grid.

The problem is, that since our interest rates $r_{1,2}$ and $r_{2,3}$ will have positive correlation: After r^u comes r^{uu} or r^{ud} which are higher than r^{dd} and r^{ud} which comes after r^d. Hence,

$$
(11) \quad E\left((1 + r)(1 + r_{1,2})(1 + r_{2,3})\right) > (1 + r)(1 + E(r_{1,2}))(1 + E(r_{2,3}))
$$

since we ignore the positive covariance. By using (9) and (10) we see that (11) can only hold if $E(r_{1,2}) < f_{1,2}$ and/or $E(r_{2,3}) < f_{2,3}$. Which is hopefully what you saw in the exercise. Note that, $E(r_{1,2}) = f_{1,2}$ will generally hold, since r^u and r^d were constructed $E((1 + n_1)(1 + r_{1,2})) = (1 + n_2)^2$, and since there is only one stochastic variable, $r_{1,2}$, there are no covariance to ignore.

5.   PLAYING A LITTLE WITH THE VOLATILITY ASSUMPTIONS, (OPTIONAL)

Maybe you think that the simple volatility structure chosen above is too simple. A lot of authors do, and they have proposed a lot of different models which all imply a specific structure on how volatility parameters depending on which state

we are in. We will not consider these proposals since they will involve a lot of theory. However, I think it would be nice to play around a little with the volatility factor.

When we use the volatility parameter σ in restriction (6) to determine r^{uu}, r^{ud}, and r^{dd}, the interpretation of σ is the same as when we use σ in the two-period grid to determine r^u and r^d: Given the state we are in, σ is the volatility of next periods one-period rate. We say that the *local volatility* is the same all over the grid.

Another approach would be the following, (we stick to the three period setup to simplify). Assume that $r_{2,3}$ will be distributed like

$$(12) \qquad r_{2,3} = \bar{r}_{2,3} e^{\epsilon_2},$$

where $\bar{r}_{2,3}$ is the expected value of $r_{2,3}$. Now, we would like to determine the volatility parameter σ_2 which is used to determine the distance between r^{uu}, r^{ud}, and r^{dd} such that the volatility of ϵ_2 match the precision with which we feel we can predict the interest rate two periods into the future. Let this volatility, the standard deviation of ϵ_2 when predicted from $t = 0$, be denoted with $\sigma_{0,2}$ Notice the difference between σ_2 and $\sigma_{0,2}$. The first is a local parameter which tells something about one-period uncertainty. The latter tells something about the uncertainty over longer time span, (in this case two periods).

In the data-sheet of my solution and in Figure 8, you will find my shaky attempt to determine the precision by which we can predict the future one-year US interest rate. The interpretation of the Figure is this: Next years realized interest rate will deviate from my prediction with at prediction error with a standard error approximately equal to 33% of the predicted value. If we try to predict the interest rate two years ahead, we should expect a standard deviation equal to 46%. Intuitively, this result makes sense. The longer horizon, the lower precision.

As you can see, you should not trust the figures too much as they appear to be subject to sizable estimation errors. For instance, it seems unrealistic that we can predict interest rates 5 years ahead better then 3 years ahead.

One interesting feature should be noticed, however. At some point in time, the volatility in the predictions seems to stabilize. The intuition is this: If you try to predict interest rates more than 10 years into the future, the observed interest rate today will not help anything. You might as well predict the long term average and expect a volatility of your prediction errors to be around 53%. I'm not sure you would get that result with a constant local volatility parameter. I.e., we might take the plot as evidence that the local volatility parameter should decrease somewhat, as longer and longer horizons are considered.

IMPLEMENTATION

Now that we know that distributions of interest rates at time $t = 2$ should have an overall volatility of $\sigma_{0,2} = 0.46 \sim 46\%$, we might start wondering how we could implement this information. Note that we will not change the simplifying structure of the restrictions in (6). But now, we would like to choose σ_2 such that $\sigma_{0,2} = 0.46$.
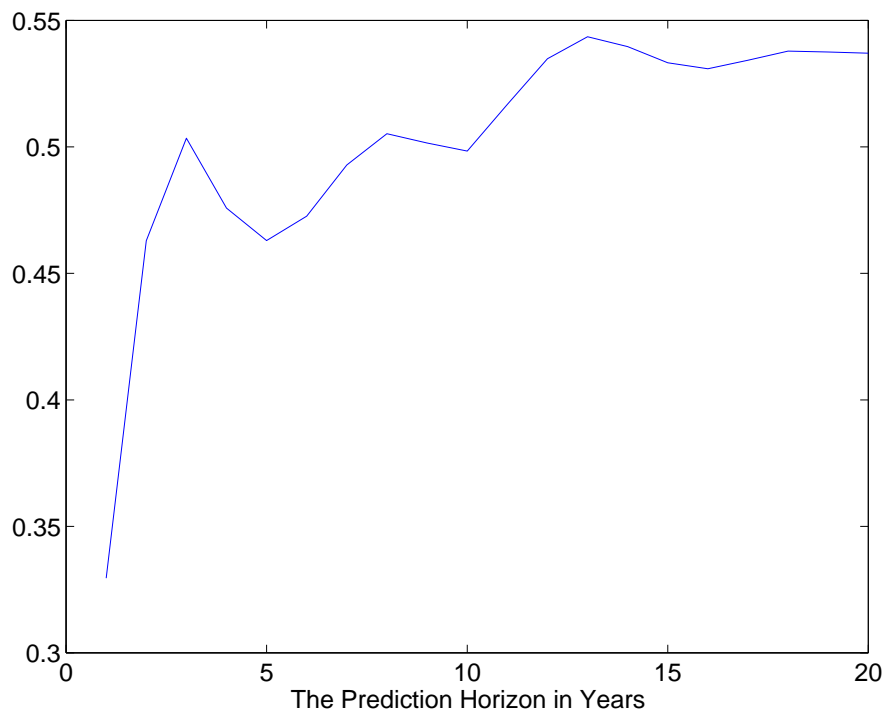
Figure 8: Volatility of predictions for one year US treasury rates.
The following autoregressive model were used to predict $\bar{r}_{t,t+1}$: $\bar{r}_{t,t+1} = \bar{r} + \alpha^t (r - \bar{r})$, where t is the prediction horizon, $\bar{r}$ is fixed at the sample average of interest rates, and r is the observed rate. The decay-parameter were estimated with least square. The volatility $\sigma_{0,t}$ is estimated as sample standard deviations of the prediction errors measured as $\epsilon_t = \log(r_t/\bar{r}_{t,t+1})$. Sample used: yearly observations for 1963-2003 used for estimation. The volatility is calculated based on predictions for 1983-2003

First, we should remember from the formulation in (12), that the volatility parameter is the standard deviation of the shock ϵ, not the interest rate. To determine the standard deviation of the shocks, we need to identify them. And therefore, we need $\bar{r}$. Fortunately, $\bar{r}$ is easily calculated:

$$\bar{r} = \tfrac{1}{4}r^{uu} + \tfrac{1}{2}r^{ud} + \tfrac{1}{4}r^{dd}$$
$$= \tfrac{1}{4}e^{4\sigma_2}r^{dd} + \tfrac{1}{2}e^{2\sigma_2}r^{dd} + \tfrac{1}{4}r^{dd}.$$

Then, the shocks are easily identified,

$$\epsilon^{uu} = \log(r^{uu}/\bar{r}), \qquad \epsilon^{ud} = \log(r^{ud}/\bar{r}), \qquad \epsilon^{dd} = \log(r^{dd}/\bar{r})$$

or

$$\epsilon^{uu} = \log(e^{4\sigma_2}r^{dd}/\bar{r}), \quad \epsilon^{ud} = \log(e^{2\sigma_2}r^{dd}/\bar{r}), \quad \epsilon^{dd} = \log(r^{dd}/\bar{r}).$$

And since the probability of each state is known by assumption, the mean value of the shocks $\bar{\epsilon}$ is also easily determined:

$$\bar{\epsilon} = \tfrac{1}{4}\epsilon^{uu} + \tfrac{1}{2}\epsilon^{ud} + \tfrac{1}{4}\epsilon^{dd}.$$

We are now ready to set up the restriction on the standard deviation of the shocks at $t = 2$:

$$(13) \qquad \sigma_{0,2} = \sqrt{\tfrac{1}{4}(\epsilon^{uu} - \bar{\epsilon})^2 + \tfrac{1}{2}(\epsilon^{ud} - \bar{\epsilon})^2 + \tfrac{1}{4}(\epsilon^{dd} - \bar{\epsilon})^2}.$$

If we substitute r^{uu} and r^{ud} with $e^{4\sigma_2}r^{dd}$ and $e^{2\sigma_2}r^{dd}$, the restriction in (13) has only two unknown, σ_2 and r^{dd}. As before, we will determine r^{dd} with the restriction in (7) when we know σ_2. But we cannot determine σ_2 from (13) before we know r^{dd} ! Hence, we have to ask the Solver in Excel to solve both problems simultaneously. To do that, we need a little trick.

SOLVING MANY EQUATION IN ONE STEP

Unfortunately, the Solver in Excel cannot handle more than one target cell. However, with a little trick, you can get the Solver to solve more than one equation at a time. We need it for the problem above, but it is of course a very general problem.

Assume that we want to solve the problem above and let e_1 be the violation of the restriction in (7) and let e_2 be the violation of the restriction in (13). If you ask Excel's solver to minimize the sum of squared errors,

$$(14) \qquad (e_1)^2 + (e_2)^2,$$

the lowest possible value is zero, which will only happen if both e_1 and e_2 are zero at the same time. Unfortunately, the function is very flat around the solution, and the solver might have a hard time finding the optimum. To get better solutions, I suggest you push the "options" button in the Solver interface, and change "Derivatives" from "Forward" to "Central". Sometimes, "Use automatic scaling" can also

help. With these changes, it might help to re-run the Solver after you have found
the first solution. Consider also the following trick:

$$100000000 \left((e_{1,2})^2 + (e_{2,3})^2 \right).$$

Simply multiplying the objective with a high number. This primitive transforma-
tion might also help, since small errors are amplified. (You might get the same
effect by increasing the precision in the Solver option interface.)

> **Exercise 14.5, (Optional):** Consider the same setup in as in Exer-
> cise 14.2.a. We will reuse r, r^u, and r^d.
>
> - Determine the local volatility parameter σ_2 for r^{uu}, r^{ud}, and r^{dd}
>   such that the overall volatility of ϵ_2 at time $t = 2$ is equal to the
>   estimate of $0.46 \sim 46\%$ while at the same time calibrating the level
>   (i.e. r^{dd}) to the term structure, (i.e. n_3).
>
> *Note: In order to impose both the volatility restriction* (13) *and the
> forward rate restriction and the level restriction* (7) *simultaneously, you
> should use the trick in* (14) *to get only one target cell.*